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FIRST MNEMONICAL LESSONS IN GEOMETRY, ALGEBRA, AND TRIGONOMETRY.

BY THE

REV. THOMAS PENYNGTON KIRKMAN, M.A.
RECTOR OF CROFT WITH SOUTHWORTH.

JOHN WEALE.

Florian Cajori.
J.H.C.

FIRST
MNEMONICAL LESSONS.

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M.DCCC.LII.

TO

SIR JOHN BLUNDEN, BARONET,

THIS LITTLE MEMORIAL

OF DAYS PLEASANTLY AND THANKFULLY REMEMBERED,

IS INSCRIBED,

WITH FEELINGS OF HIGH AND MOST

DESERVED ESTEEM,

BY HIS FAITHFUL SERVANT,

AND SINCERE FRIEND,

THE AUTHOR.

PREFACE.

It is reckoned a small ambition which is content to write a book of rudiments; and a wise man will hardly do this, unless he knows beforehand, from his fame or scholastic influence, that the profit of the work will compensate its lack of honour. With this reflection, I have 'kept my peace' *above* 'ten years;' and, finally, misdoubting the profit which my obscurity might command, I have saved my paternal feelings, by presenting this little book to the publisher as an addition to his cheap series of elementary mathematics. To this I was moved by my admiration of the remarkable zeal and spirit displayed by him for the diffusion of scientific knowledge. While it is hoped, that these pages will be easily understood by readers who are familiar with arithmetic of whole numbers and fractions, and with the extraction of the square root, it is evident, from the arrangement and treatment of the topics, and particularly from the paucity of examples, that this little work is intended not as a substitute, but as a companion, for other rudimentary treatises. If the experience of others in tuition agrees with my own, I may perhaps look to reap a little praise—not mathematical, on ground like this, but simply didactical—the praise of teaching well, of which I confess myself covetous. It appears to me, that distaste for mathematical study often springs, not so much from any abstruseness in the subject at any point, to the student who has mastered the approaches, as from

the difficulty generally felt in retaining the previous results and reasoning. This difficulty is closely connected with the *unpronounceableness of formulæ:* the memory of the tongue and of the ear are not easily turned to account: nearly everything depends on the thinking faculty, or on the practice of the eye alone. Hence many, who see hardly anything formidable in the study of a language, look upon mathematical acquirements as beyond their power, when in truth they are very far from being so. My object is to enable the learner to *talk to himself,* in rapid, rigorous, and suggestive syllables, about the matters which he must digest and remember. I have sought to bring the memory of the vocal organs and of the ear to the assistance of the reasoning faculty, and have never scrupled to sacrifice either good grammar or good English, in order to secure the requisites for a useful *Mnemonic,* which are smoothness, condensation, and jingle. I would beg to have judgment pronounced upon my method, not from its usefulness or beauty in the eyes of a mathematician, but from its success, good or ill, in the instruction of young persons, of ordinary apprehension, who have all their mathematics yet to learn.

My only apology for the form and colloquial style of these lessons is *the fact,* that they were at first begun in their present shape, save a few trifling variations, for a juvenile class, which included certain nieces of mine. The readiness, with which school-girls of fair capacity, who had been well taught arithmetic, apprehended and retained the subject by these aids, strengthened an impression, which I had cherished for many years, that something of the kind might be generally useful. If my method finds favour with students, it will be easy for me to extend the assistance, here offered

at the entrance, to the subsequent and higher stages of their mathematical career; for I have good store of such aids, adapted to most of the leading topics in mathematics, both pure and applied. See a Paper "On Mnemonical Aids in the Study of Analysis," in the Ninth Volume (N. S.) of the *Memoirs of the Manchester Philosophic and Literary Society.*

The art and mystery of Mnemonics has been brought into disrepute by such writers as Feinagle and Coglan, men worthy of chairs in the university of Laputa. Their cumbrous inventions are about as fit to be compared, for elegance and speed, with the ἔπεα πτερόεντα of Richard Grey, as a Dutchman's ox-wagon in Kaffirland with a nobleman's chariot in Middlesex.

Concerning Grey's *Memoria Technica* there are two opinions; one, of those who *in their student-days* had the good fortune to have the book placed in their hands; and another, of those who have learned (and forgotten?) their chronology, &c. without it. I am sometimes amused by the readiness of the latter division of persons to pronounce judgment on the philosophic old Doctor, with the air of men who have well considered the matter. More than one good scholar and good teacher do I know, who dispose of him coolly thus: 'the difference between studying and not studying Grey's book is this, that, in the latter case, you have certain things to learn and remember, and, in the former, you have the same things to learn, and a mass of frightful jargon besides.'

Once upon a time, there was a handy man, who took a fancy to joinering. He went up the town, and bought a complete assortment of carpenter's tools, everything from a wood-man's axe to a sprigbit. As he was scratching his ear in meditation about the best way to convey them home, a simple bystander suggested, 'Why don't

you look out for a wheelbarrow?' 'Because I am not an ass,' was the curt reply: then, softening a little, he added, 'Do you see, my good friend, the difference is exactly here: as it is, I have my tools to carry home; if I took your advice, I should be saddled with both the tools and the wheelbarrow.'

CONTENTS.

MNEMONICAL LINES.

[1], [3], &c., mark the Mnemonics; (1), (5), &c., the Sections or Articles in which they occur.

ART.

[1] (1) Líne cutting páralls. makes álter. ints. équal, and
éxt. equal ínt. op.
And cútter cuts paralls., if equal altér. ints., or
ext. equal int. op. Prop. B.

[2] ... A párgram. has equal op. ángs., or op. sídes.
'Tis párgram., if équal op. ángs., or op. sides.

[3] (5) Like sign's give plús,
Unlike mĭnús,
For sign of pro. or quo.

[4] (6) ēx is ý gives óri. lí.

[5] (7) The line (Dif. ȳb is ēx) is párl. to (y is ēx).
pron. yb, *wybe.*

[5′] (8) One 's `vi (xl) and vi. (yà) pron. *vixle.*
From `Or. cuts l and à.

[6] (13) If páralls. cut légs,
vi. ségs. is vi. ségs.,
And ví. (parall. cútters) is ví. (corre. ségs.):
Measure the segs. from meet. of legs.

[7] (14) Qua. póth is bóth qua. sides.

[8] (15) pér. on póth. is méan segs;
and síde is méa. (poth. nígh seg.)

[9] (17) vĭD(y's) is vĭD(x's), pron. *vidwise.*
with 10 o'er 12—déxes: read ten o'er twelve.
tĕr x'Dĭ(y's) is níl, pron. *Diwise.*
at ó12 roúnd, for gíl. ó12 read *ówe untwo.*

ART.

[10] (18) *C*'s *Ax* and *By*;
Dot second li:
Dĭ(*CA*)'s bў Dĭ. (*BA*)'s, (dŏt oùts.), is mèeting *Y*.

[11] (19) After minus untyin'
Change every sign.

[12] (21) Pŏth.{Dí(*xl*) Dĭ(*yá*)}, — pron. dixle. (*ya*) a monosyl.
joins(*x'y*) to (*la*), — *xy* a dissyl.
in co-ords. rĕctā.

[13] ... *DUQ*{Dí(*xl*) Dĭ.(*yá*)} — pron. dixle; *ya*, one syl.
Is *r r'*, (in Recta) — *rr'* and *la* both dissyll.
Gives círc. cĕn (*lá*):

[14] (22) QuăSŏrD(*áb*) is DUQ(*áb*) mol two(*áb*): — *ab* one syl.
Súm(*bă*).Dĭ.(*bá*) is Sq'.*b* le sq. *á*. — pron. *squibble squa.*

[15] (24) If *sq.y'* le twŏ *yá* be *N*, — pron. *squi.*
y's *á* mol RóoM *ăsq N'*, — pron. *ask.*

[16] (26) If nil be (sq. *y'* le *py'* and *q'*) — pron. *squi.*
p's sùm, and *q* is prod. of roo.

[17] (29) DUQ(*y'x*) is Ty' — *yx*, pron. *wix.*
Has tán. nil's y':
a. Rad per'p on chór.'s biChór:
b. Toŭch-r'ad is per' on tán:

[18] (30) If á's b, An'g is Báng;
and per'c is bic' is bĭCáng.

[19] (31) Rim-angle on arc is arc-démi,
And right is the angle in semi.

[20] ... (Seg . S,ég) in *P* – ríng,
Is mŏl SU'D (*poc* wi'ng),
For *P* oút or in ring. — Pron. *pout.*

ART.

[21] (33) x and y are Cos and Sī
ŏf Scăle-a'rc from OX; cĕn. o'r. Ax Rīgh.

[22] ... So'rCo.Po'th is o'p or ad; H.
Sī bў Cŏ's ta'n, is vi(ŏp. a'd). K.

[23] ... Co'rSin ω''s SĭnŏrC(ri'ght min ω') G. ω, pron. ō.
Co'rS is le'm the Co'rS of – ple'm L.
CorS(π et ω) is le' CorSω. L'. pron. pĕt$\acute{\omega}$.

[24] ... Tăn Cos Sĭn are rec.
Cŏtă Sec Cŏse'c.

[25] ... *DUQ*(Sĭ Cŏ)'s One: (M)
Co right is none
Co none is one.

[26] (34) Draw pérc: SUD (bá) is SŬD (ségs of c').
And sq'.b is DUQ ác mol (ség. op) twō c, A.
sq'.b. pron. *squib.*
As 'tùse or 'cùte is Báng op. b;
Or, sq'.b is DUQ ác le *C*oBáng two ác. B.

[27] (35) Sĭn ($\acute{a}$ mŏl τ) ĭs S$\acute{a}$.Coτ mŏl C$\acute{a}$.Siτ; pron. a, τ, as a, t.
Cŏs ($\acute{a}$ mol τ) is C$\acute{a}$.Coτ lĕm S$\acute{a}$.Siτ. mol τ, one syl.
c pérc is *ba* Sĭ Cáng;
then pút (α mól τ) for Cang;
(new—óld a) máke right áng.

[28] (36) HaS (ab) mol HaD (ab) is a or b;

[29] ... SíM mol SíD's two SórC.CorS,
CóD mol CóM's two CórS.CorS.

[30] ... S$\acute{a}$ mol Síτ's two SórCHaM.CórSHaD,
C$\acute{a}$ mŏl Cŏτ's mól twŏ CŏrSHáM.CŏrSHáD.
SH and CH pron. as in *Cheshire.*

[31] ... CórS is SU'D or twó CHa.SH.

ART.

[32] (36) RoóP slĕb.š.sléc.slá,
Is Síne Bang hálf cá,
Is CHáSHca, is Area. Pron. CH and SH as in *Cheshire*
Write 'sqb' in quaCH and quaSHaBang,
By 'CorS is SUD'...and 'DUQ Si...'

[33] (37) Tásquaf A′ is slĕb. sléc by š.slá.

[34] ... Síde to Sinóp as síde to Sinóp.

[35] ... SŭbўD(Sínes or Sídes) is tăf Súm by tăf Díff.

[36] ... tăn(ᾰ mólῶ)'s tᾰ mól tω bў DórS (ŭn tᾰ.tώ).

[37] (41) DUQ {bic (half *c*)}'s half DUQ {*ab*}.

[38] (42) E. BiCáng is RoóFDuP (*áb*, segs),
C. Is CósHaCáng of HárM legs;
And ví (*ab*) is the ví (segs).
sH, pron. as in *cash* ; of for *times*.

[39] ... HarM (*áb*) is twŏ *áb* by S (*áb*).

[40] ... In quad. inscri. two opps. are π, π pron. *pi*.
And pró digs is DuPóppo.si. I.

[41] (43) J. Dím. out círc is *ba* to pérc.
Z. In. rád. of sémp. is Ar′e, of is ×.
K. Four out. is *bac* by Are.

[41′] ... RooP. ín. e. ráds is Are;

[42] (44) In símils, quŏt. Ares
Is vi(lik.si. squares) pron. villiksy.

[43] (45) c. Táf. Sin is ver.;
b. Two quásif is ver.

[43]′ ... Rim's twó. π . rad ; Are's r' . π . rad.
pron. twopyrád, r.pyrád.

ART.

[44] (46) (á to lĕ p′) is cĭp (á to p′).

[45] ... ĕ tó vĭ(twŏ threé), pron. tovvy.
is Croot of sq.e,
is squared Croot é.

[46] (47) Prŏd. ány pows. of x, is
x to (sum of -dexes).

[47] ... a. $\begin{cases} c \text{ tód } a \text{ tód's } c\breve{a} \text{ tód,} \\ c \text{ tod by } a \text{ tod's quo. tod.} \end{cases}$
b. $\breve{n}$ tŏ více $\breve{a}$ tŏ víde is the e^{th} root of n' toc a tód;
c. Pór Quo. pǿws is power of PorQ,
d. Pór Quo. roóts is root of PorQ.

[48] (48) a's your báse to (fít log a)
a is ten to cóm. log a.
a. lóg t and lóg a's lóg (ta),
a′. (lé log a for ví(ta:)
b. lóg (a tôp) is p lŏg á.

[49] (50) You want to fi′, fi for find.
Dilógs or i.
and Di {lŏS(Ni) lóN} is (táb. diff.) $\acute{\imath}$:
first take i for point i.
Nég. is ló. of frac. pro.
Pow′.ten néxt sub num. is -dex.

[50] (51) The Síde c' is Cóγ Sum($\acute{a}b$),
Where Síγ meăn ($\acute{a}b$)'s CHăCáng. HarM ($\acute{a}b$).

[51] (52) Sin $\acute{\omega}$ (xy̆ lĕ ýx) at ŭn twó, pron zy le wix.
Is twíce Are (Or′ un twó).

[52] (53) Lŏr's Cón. Sin ω by bás(fics ω):
Is Cón. cĭpŏth fics, if right is ω.

[53] (54) văl8′. vĭ LŏrCón is pér lĭn. ŭpón.

ART.

[54] (55) a, Síβ to SĭD($\omega\beta$), or táβ in RAx, pron. Dōb.
Is *é* in *ў*'s *éx* : β ŭp ríght LĭnAx.
b, tăβ's *é* Sin *ώ* bў S(ŭn *é* Cos *ώ*).

[55] ... e, băs[Dí(xl) *ω* D(áy)] joins xў to lá,
xy a dissyl. Di(xl) pron. dixle.
c, LiL Díβ's you knów by 'tăn(á mol *ώ*).'

[56] (56) gĭl's lé cotá.β is pérlĭn's taβ.

[57] (57) a Cŏs A′ng's b Cŏ.Bang,
a Sin Ang's b SiBang,
b's a, (in Rom ba) ăre pĕrc bíc ănd biCang. [v. 18]

[58] (58) P...Ang cŭts *a′* ĭn dŏtA′, &c.,
*AB′.BC′.CA″*s *AC′.CB′.BA′* ; dŏt áltĕrnā′
Then for lines put oppo. sines.
pron. ăbbĭccá, ăckĭbbá.

[59] (59) Dĭ(xē)'s ϕ. Dĭ(yí) is soúghl thrŏ′ ĕí ; ϕ pron. phi.
Nil's vlĕϕú is soúghl throŭgh(vú) ; ϕu pron. fu.
(y le ϕ)'s ēx, soŭghl párl to y's ēx.

[60] (60) If ábc...fórth ĭs Ari.Se :
Ult and a is pĕnúlt. and b ;
Sum Ari.'s hălf n.Sŭm(áz),
and (d lĕ dn) is Dif.(áz). *dn* a dissyl.

[60′] ... cips Ari.Se are Harmo.Se.

[61] (61) G. { z's á of (ĕ tos); nŭm. térm. is t ; of = times.
ă lĕ zé by D(ŭné) is Sum Geo. Se. }
H. One bў D(ŭné) ĭs fr8 ŏn é,
is Súm, ĭf ĕ's frác. of an infi. Se.

[62] (62) If p ĕléms. hăve m ă's, é b's, i c's,
The perms. in p's are p fags bў (m fags. e fags.
i fags).

ART.

[63] (64) Comb. nón-repéa.'s of *n* in *d*'s,
Are *d* *n*-bácks by *d* fags.

[64] ... The répe. combs. of *n* in *d*'s,
Are *d* *n*-úps by *d* fags.

[65] (65) Non-repe. vars. of n in d's
Are d n-backs.

[66] (65) The répe. várs. of n in p's
Are n to p^{th}.

[67] (67) Sŭtón(ŭn r)? wrĭte fr8 ŏn r;
Then r to í you multiply
By (i n-bácks bў i fags):
If n hăs dĕn.é,
Put r vĭ(rĕ); top dits wĕd e.

[68] (70) Roŭnd ev. or o. with *pin* or no
For signs you go.

SYMBOLS AND ABBREVIATIONS.

$+$ is the sign of addition; *plus.*

$-$ .... of subtraction; *mĭnus.*

$=$ .... of equality; *equals:* sometimes, *which equals.*

$3+2-1=4$ is read, 3 plus 2 minus 1 equals 4.

$\times$ is the sign of multiplication; *times:* a point is often used instead of $\times$.

$\div$ or : is that of division, as

$12:(3+1) \doteq 12 \div 4 = 3$; $a:(x+y)$ is a divided by $(x+y)$.

$>$ means *is greater than,* sometimes *greater than.*

$<$ means *is less than,* sometimes *less than.*

$\perp$ stands for *perpendicular*, or *perpendicular on.*

Between algebraic quantities written close together $\times$ is always understood; thus $(5-2)(6+1)$ is 3 times 7; $(a+b)(x+y)$ is $(a+b)$ times $(x+y)$; xy is x times y.

Quantities collected by a tie or vinculum are not to be treated as if they were untied. The whole tied quantity is affected at once by $\times$ preceding or following; as $(5-2) \times (6+1)$ above. But

$$(5-2)\times 6+1 = 18+1 = 19,$$
$$5-2\,(6+1) = 5-14 = -9.$$

CORRIGENDA.

PAGE	LINE	
7	5	exchange x and y.
7	19	for $X''OY''$ read $X''O'Y''$.
10	3	for q or n read p or m in this locus.
12	12	for x_1 read x.
15	last line,	for + read −.
20	8	from bottom, after BC, insert [1].
22	16	for $A.B$ read AB.
35	11	for m_1q_1 read p_1q_1.
...	12	for q_3q_3 read p_3q_3.
32	7	from bottom, for $Cf = cf'$ read $cf = c'f'$.
65		suppress the misplaced enunciation D.
95	10	for BB read CB.
...	6	from bottom, for *of* read *is*.
96	12	for *same triangles* read *same angles*.

140, 141, and 143, for [58], [59], and [60], read [57], [58], and [59].

The reader is requested to begin by making these corrections.

TO THE YOUNG READER.

If you think of four numbers, calling them a, b, c, d, and tell me nothing about them, more than that the first is double the second, and the third is three times the fourth, I can write down what I know as follows:

$$a = 2b, \qquad a \text{ is equal to twice } b,$$

$$c = 3d, \qquad c \text{ is equal to thrice } d,$$

and from these two assertions I can draw various conclusions: thus,

$a + c = 2b + 3d$, by addition of equals,

and $a - c = 2b - 3d$, by subtraction of equals,

and $a \times c = 2b \times 3d$, by multiplication of equals,

read a times c equal $2b$ times $3d$;

also, $\frac{a}{c} = \frac{2b}{3d}$, by division of equals,

a divided by c equals $2b$ divided by $3d$.

Hence follows, if I multiply these last equals by c,

$$a = \frac{2bc}{3d},$$

i.e. a = twice b times c, divided by 3 times d; and from this, multiplying by $3d$, I deduce

$$3ad = 2bc,$$

thrice a times d is twice b times c.

All these equations must be true, if the first pair are true. Suppose now that I have the information that the product of a and d is 10, without knowing whether these

are whole numbers or fractions; and that I am told in addition, that b and c are whole numbers, and b greater than c. The last equation shews me that $30 = 2bc$, whence it follows that $15 = bc$; from this I gather that c is either 3 or 1; because 15 cannot be the product of any whole numbers, except the pairs 5 and 3, and 15 and 1.

All this is premised merely to shew the young reader, to whom algebraic characters are new, that we can often reason with symbols of unknown quantities, and, by easy arithmetical operations upon them, arrive at conclusions that may greatly increase our knowledge.

The reader is supposed to know the meaning, and to see the truth of, such assertions as these:

$$5 - 2 = 5 + (-2) = 5 - (+2) = 3,$$

$$5 - (-2) = 5 + 2 = 7;$$

$$a - b = a + (-b) = a - (+b),$$

$$a - (-b) = a + b.$$

See the conversations at the end of this little volume, which are intended to be read by the beginner, as notes on the opening lessons.

FIRST MNEMONICAL LESSONS

IN

GEOMETRY, ALGEBRA, AND TRIGONOMETRY.

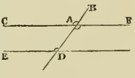
LESSON I.

Uncle Penyngton, Jane, Richard.

Uncle Penyngton.

1. You are now, my dear children, familiar with the four rules of arithmetic in whole numbers and fractions, and with the extraction of the square root; and Richard can prove, from the first book of Euclid, the propositions following:

Prop. A. Two intersecting straight lines, (AD and CF) make either four *equal* angles, (all *right* angles), or a pair of *acute* angles, (each *less* than a right angle) and a pair of *obtuse* ones, (each *greater* than a right angle). The two acute angles are vertically opposite and equal, and so are also the two obtuse ones; and any unequal pair, an acute with an obtuse angle, make two right angles.



Prop. B. A line cutting a pair of parallels, (lines which can never be produced to meet), makes the exterior angle (BAC) equal to the opposite interior (ADE), and two alternate interior angles, (FAD, ADE) equal to each other: and, conversely, if a line cut two lines so as to make either the two alternate interior angles equal, or the exterior angle equal to the interior and opposite, these two lines are parallel to each other, and can never meet.

opposite, on the other parallel; *interior*, within the parallels; *alternate*, on different sides of the cutting line.

Prop. C. A parallelogram, (which is defined to be a four-sided figure whose opposite sides are parallel lines), has

its opposite sides equal, and its opposite angles equal: and conversely, if a quadrilateral, (i.e. a four-sided figure) has either two pairs of equal opposite sides, or two pairs of equal opposite angles, the figure is a *par*allelo*gram*.

Prop. D. Any exterior angle (CBD) of a triangle is equal to the sum of the two interior remote, (CAB and ACB), and the three interior angles are together equal to two right angles, i.e. to an exterior with its interior.

Observe that when an angle is denoted by three letters, the one standing at the angular point is placed in the middle. ABC, BCA, CAB are the three angles B, C, and A of the triangle.

As Jane has never learned Euclid, she may for the present take it for granted that these propositions are true. They are so nearly self-evident, that it is one of the most difficult things to establish the truth of them all by *rigorous* demonstration; and neither Euclid, nor any other geometer, has done this by arguments that you could at present comprehend. An *assumption* of some kind is always found necessary, which requires demonstration as much as the propositions to be proved thereby.

Jane:—It appears to me that the propositions are much easier to believe and comprehend than to remember.

Uncle Pen.:—You know, from good old Dr Richard Grey, the value of contraction and cadence as aids to memory; and I recommend you strongly, *before you begin* to try to remember these properties, to learn perfectly by heart the following mnemonical aids, referring to Props. B and C, and to teach them to your ear and to your tongue, each of which has a memory of its own, by saying them again and again with a sing-song repetition, marking well the accented syllables.

[1] Líne cutting páralls. makes álter. ints. équal,
and éxt. equal ínt. op. Prop. B,
And cutter cuts paralls., if equal altér. ínts.,
or ext. equal int. op.

[2] A párgram. has equal op. ángs., or op. sídes.
'Tis pargram., if equal op. angs., or op. sides. Prop. C.

paralls for parallels; alter. ints. for alternate interiors; ext. for external; op. for opposite. Observe that in 1. or Prop. B, there are four ext. angles, and 2 pairs of alter. ints. viz. a pair of equal acute, and a pair of equal obtuse angles.

I shall now proceed, according to a promise I recently

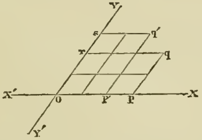
gave you, to show you the easiest way, (for there is no little difference as to difficulty in the approaches) to the delightful fields of geometry, through which alone you can be introduced to the sublime regions of Mechanical Science, terrestrial and celestial. And the easiest way in this instance is the shortest way.

First of all, we must express positions of points by numbers, and thus reduce questions about points, lines, and areas, to questions of arithmetic. Suppose a group of points before you, how would you contrive this?

Richard:—It would be easy to number them one, two, three, &c.; would that do? By this plan we should have a number for every point, and should know what we were talking about.

Uncle Pen.:—You might thus *name* your points arbitrarily; but we must have a system, by which all numbers shall determine certain points. I am about to show the happy contrivance of the celebrated Des Cartes, which *fixes* the position of a point by a pair of numbers.

2. Let two leading lines be chosen in the plane of the paper, supposed unlimited in every direction, which meet in a point O, called the *origin*. XOX' is the axis of x, and YOY' the axis of y.

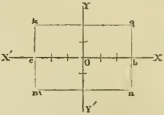


X' is read X dashed.
Y' is Y dashed.

If we take a certain length for our unit, and call it an inch, and measure from O along OX, say 3 inches to p, and from p in a direction parallel to OY, say 2 inches to q, we have the point ($x=3$, $y=2$), or, more briefly, the point (3, 2), viz. the point q. Op is the x, and pq is the y, of this point. The point q', whose x and y are 2 inches and 3 inches, is the point ($x=2$, $y=3$), or (2, 3). The x and y of a point are called its co-ordinates, and the leading lines OX and OY are called axes of co-ordinates, or co-ordinate axes. If we draw qr parallel to the axis of x, we may call Or, which by [2] is equal to pq, the y of the point q, and qr [2] the x of it. Thus having given us the co-ordinates

$x=3$ and $y=2$ in inches, we can find by measurement the position of the point $(3, 2)$, or q: and if this point had been given us in position, we could have found the co-ordinates, by simply drawing from q a parallel to OY, and then measuring the inches in Op and pq. I shall *assume* once for all that we have the power of joining any two given points by a line, of drawing a parallel to any line, and of measuring lengths to any degree of accuracy, say to the millionth part of an inch, if required.

Jane:—This is very ingenious of Monsieur Des Cartes; but I do not yet see how confusion is always to be avoided about the point $(3, 2)$. If the axes were chosen at right angles as thus, (or indeed at any angle), I cannot see what right q has to be considered the point $(3, 2)$ rather than k or m, or n; all which are alike determined by three inches from O on the axis of x, and by two inches on that of y.



Uncle Pen.:—Your objection is well stated. The points q, k, m, n, are called $(x=3, y=2)$ or $(3, 2)$, $(x=-3, y=2)$ or $(-3, 2)$, $(x=-3, y=-2)$ or $(-3, -2)$, $(x=3, y=-2)$ or $(3, -2)$. The directions OX and OY are positive or above nothing: their opposites OX' and OY' are negative or below nothing. The length -3 (read minus 3) is 3 negative, or below nothing: and the difference between 3 and -3 is, that the first denotes a length measured, from whatever point, or on whichever axis, in the *positive*, while the latter is a length backwards, in the *negative* direction. If a point moves by an inch at a step from b to c, its distances, measured from 0 or zero, will be after the successive steps, from rest at b, $3-1=2$, $3-2=1$, $3-3=0$, $3-4=-1$, $3-5=-2$, $3-6=-3$; i.e. it will have a distance from 0 after the fifth and sixth steps, -2 and -3, or 2 and 3 negative, below nothing.

It is of no consequence, when we choose our axes, which direction along either we make the positive one; but we shall *in general* consider OX to the right, and OY upwards, to be the + (plus) or positive, and their opposites to be the − (minus) or negative directions. Observe that the x of a point is often called its *abscissa* and the y of it its *ordinate*.

3. You know, when our axes and our unit length are chosen, what point is given by the two statements,

$x = 3$, read x equals 3,

$y = 2$, y equals 2.

From this pair of assertions, or *equations*, follows the equation

$\frac{x}{y} = \frac{3}{2}$, . read x by y equals 3 by 2; *divided* by.

Consider now the points

$x_1 = \frac{3}{2}$	$x_2 = \frac{3}{4}$	$x_3 = \frac{3}{8}$	$x_4 = \frac{3}{16}$	$x_5 = \cdot 6$	$x_6 = \cdot 9$	&c.
$y_1 = 1$	$y_2 = \frac{1}{2}$	$y_3 = \frac{1}{4}$	$y_4 = \frac{1}{8}$	$y_5 = \cdot 4$	$y_6 = \cdot 6$	

Read $x_1\ y_1$, x at 1, y at 1; $x_2\ y_2$, x at 2, y at 2, &c.; these *subindices* serve merely to show to the eye that x and y, x_1 and y_1, &c., are co-ordinates of different points; $x_a\ y_a$, $x_b\ y_b$, (x at a, y at a, x at b, y at b, &c.) would answer the purpose as well. From these six pairs of equations follow in order by division of equals by equals,

$$\frac{x_1}{y_1} = \frac{3}{2};\quad \frac{x_2}{y_2} = \frac{\frac{3}{4}}{\frac{1}{2}} = \frac{3}{2};\quad \frac{x_3}{y_3} = \frac{\frac{3}{8}}{\frac{1}{4}} = \frac{3}{2};\qquad \frac{\frac{3}{4}}{\frac{1}{2}} = \frac{3}{4} \div \frac{1}{2} = \frac{3}{4} \times \frac{2}{1} = \frac{3}{2}.$$

$$\frac{x_4}{y_4} = \frac{\frac{3}{16}}{\frac{1}{8}} = \frac{3}{2};\quad \frac{x_5}{y_5} = \frac{0.6}{0.4} = \frac{3}{2};\quad \frac{x_6}{y_6} = \frac{0.9}{0.6} = \frac{3}{2}.$$

Of the x and y of any of these six points, all within a short distance from 0, can be affirmed, as of our first x and y,

$$(a) \qquad \frac{x}{y} = \frac{3}{2},$$

that the quotient or proportion of every pair is the same. And we can find any number of such points, as near to each other as we please. As from such an equation as $\frac{9}{6} = \frac{3}{2}$ follows, if we multiply these equals by 6×2, $9 \times 2 = 6 \times 3$; so from the above equation follows, if we multiply its equal members by $y \times 2$, (y times 2, or twice y),

$2x = 3y$, (twice x is 3 times y.

whence dividing these equals by 3,

$$(a') \qquad y = \frac{2x}{3}, \qquad y = \text{two-thirds of } x.$$

which is the same law, or condition between x and y, with (a).

Richard:—I do not see what information is conveyed by (a) or (a'); for the x and y in them have no more to do with our point (3, 2) than with any other: x and y may mean anything.

Uncle Pen.:—Not exactly. Although (a) and (a') do not state the values of x and y, they both affirm the same property; the former tells us, that if these numbers vary, they must always maintain one proportion: the latter asserts, that whatever number x may be, y must be two-thirds of it.

If we divide the equals above by 2 instead of by 3, we obtain

$$(a'') \qquad x = \frac{3}{2}y, \qquad \text{read } x \text{ is } \frac{3}{2} \text{ of } y, \text{ or } \frac{3}{2} \text{ times.}$$

the same law still.

Observe here, that when we wish to *represent* the product of a number and a symbol of a number, or that of two symbols, we write the two quantities together, $3x$, cx, or else with a point between them, as $3.x$, $c.x$. You may read these either three x, cx, or 3 times x, c times x; the latter is the *best and safest* way.

And *sometimes* the product even of two numbers is represented by writing them together with a point between them; but care should be taken to place the point in this case at the bottom between the figures, to distinguish it from the decimal point. Thus $3.5 = 3$ times $5 = 15$; but $3{\cdot}5 = 3$ point $5 = 3\frac{1}{2}$. Observe that the best way to read a decimal fraction is to call the point *point;* as 0·1, 6·01, 72·006, are, point 1, 6 point nought 1, 72 point nought nought 6.

If then we choose any length upon OX for x in (a'), the corresponding y is always $\frac{2}{3}$ of that length: x and y so found give a point in the series. Put then for x in order the numbers 0·1, 0·2, 0·3, &c. and you obtain in order for y $\frac{0.2}{3} = \frac{1}{15}$ of an inch, $\frac{2}{15}$, $\frac{1}{5}$, &c.; and thus a series of points are found pretty near to each other, viz.

$$\left(\frac{1}{10}, \frac{1}{15}\right), \left(\frac{1}{5}, \frac{2}{15}\right), \left(\frac{3}{10}, \frac{1}{5}\right), \text{ \&c.}$$

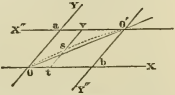
How will these look when found?

Richard:—They will form a dotted trace of some kind. I wonder, what would be its shape and direction.

Jane:—The equation $y = \frac{2x}{3}$, though it gives no information about any particular point, may perhaps tell us something about the figure formed by them all. Is it so?

Uncle Pen.:—We shall see that presently. First of all this equation (a') shows that if $y = 0$, $x = 0$ also, so that the point O, which is $(x = 0, y = 0)$ is one point in the series. The figure is therefore either a *bent* line, or a *straight* line, passing through the origin. I will prove that it is not a bent line.

4. For if it is a bent or curved line, let *any* right line OO' through the origin, which meets it again, meet it again *first* at some point O': the co-ordinates of O' can be drawn; let them be $(x = m, y = n)$. Place yourself now directly opposite to me, and taking $O'X''$ and $O'Y''$ drawn parallel but in opposite directions to OX and OY, for your positive axes of x and y, find by equation (a') referred to your axes, a series of points in the locus (a'). The angle $X''OY'' = XOY$, and my origin O will be your point $(x = m, y = n)$, by [2]; and as the conditions determining your curve and mine are exactly similar, $O'O$ will meet your curve again first in O, and your figure will appear to you as mine to me, and will be concave or convex to $O'X''$ as mine is to OX; so that the two curves will lie on opposite sides of $O'O$, and *will have no point in common* between O and O'. Let the point s, whose ordinate meets OX and OX'' in t and v, be *any* point of my series between O and O'. At the points O' and s, by (a'), I have



$$O'b = \frac{2}{3} \cdot Ob,$$

$$st = \frac{2}{3} \cdot Ot, \qquad \text{or subtracting equals from equals,}$$

$$O'b - st = \frac{2}{3} \cdot (Ob - Ot), \qquad \text{or because } O'b = tv, \text{ by [2],}$$

$$tv - st = \frac{2}{3} \cdot tb, \qquad \text{or since } tb = O'v \text{ by [2],}$$

$$sv = \frac{2}{3} \cdot O'v;$$

NOTE. *Ob*, *O'b*, *st*, &c. stand here for the *number* of linear units, or inches, in the designated lines; and we shall frequently have occasion to substitute *lines* for *numbers*, which express their lengths.

from which it appears, comparing this with (a') as employed by you, that s, *any* point in my series, is a point of your series; or the two curves *have all their points in common* between O and O'; which is absurd. This contradiction is involved in the supposition that the series of points is a curve in *any* part of its course; for to such a curve a line can be drawn from O meeting it again first in *some* point O'. Hence it is not a curve, but a straight line OO'.

You know that $(x=-3,\ y=-2)$ is a point within the angle $X'OY'$. Now as three negative inches are thrice one negative inch,

$$-3 = 3 \times -1, \quad \text{and}$$

$$-2 = 2 \times -1, \quad \text{whence by division of equal pairs,}$$

$$\frac{-3}{-2} = \frac{3 \times -1}{2 \times -1}, \quad \text{or} \quad \frac{-3}{-2} = \frac{3}{2};$$

which, compared with (a), proves that $(-3, -2)$ is a point of the series formed from (a) or (a'). And as m negative inches, whatever number, whole or fractional, m may be, is m times one negative inch,

$$\frac{-m}{-n} = \frac{m \times -1}{n \times -1} = \frac{m}{n},$$

$$\text{so that if } \frac{m}{n} = \frac{3}{2}, \ \frac{-m}{-n} = \frac{3}{2} \text{ likewise.}$$

This shows that for every point of the series, (m, n), in the angle XOY, there is a point exactly corresponding to it in position, in the opposite angle $X'OY'$, namely, $(-m, -n)$. Hence it appears that the series will have the same figure, as to the axes, in both angles, or the points form a straight line extending in opposite directions from O.

5. From $\frac{-3}{-2} = \frac{3}{2}$, it appears that *the quotient of two negative numbers is positive.* It follows, that *the product of two negative numbers is positive;* for $-3 \times -2 = -3 \div \frac{1}{-2}$, equal the quotient of two negatives. Here $\frac{1}{-2}$ cannot be positive, for if $\frac{1}{-2} = p$, a positive quantity, it follows that $1 = -2 \times p$, a negative quantity, which is absurd. We have then

$$\frac{1}{-2}=-\frac{1}{2}=\frac{-1}{2}, \text{ and } \frac{m}{-2}=-\frac{m}{2}=\frac{-m}{2};$$

$$\frac{m}{-n}=-\frac{m}{n}=\frac{-m}{n}; \quad -m\times-n=+mn=+m\times+n.$$

The product of a positive and a negative quantity is negative; for 3×-2 inches $=-6$ inches, evidently. Hence *the quotient of a positive by a negative, or of a negative by a positive is negative.* For $\frac{3}{-2}=3\times\frac{1}{-2}$; and $\frac{-3}{2}=-3\times\frac{1}{2}$; both which are products of a negative and a positive.

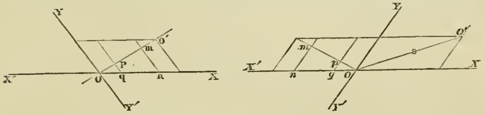
These results are easily retained by the following rule.

The *pro*duct or *quo*tient of two quantities having *like* signs (whether both positive or both negative) is *positive.*

The *pro*duct or *quo*tient of two quantities having *unlike* signs is negative. Or you may say it thus:

[3] Like sign's give plús,
Unlike mĭnús,
For sign of pro. or quo.

6. We have made no restriction as to the angle XOY,



and whatever this may be, the equation $y=\frac{2}{3}x$ represents a line drawn through O within that angle, if y and x are the co-ordinates of a point referred to the axes forming it.

Let now two other axes OX, OY, be drawn, as above, to the left of our former ones, making the angle XOY $=X'OY$; and let x and y be the co-ordinates of a point referred to these new axes, while x and y retain their signification as to the old ones. The locus

$$\frac{\mathrm{y}}{\mathrm{x}}=\frac{2}{3}, \text{ which is } \mathrm{y}=\frac{2\mathrm{x}}{3}, \qquad \text{a,}$$

is a right line; let it be OpmO′. The locus

$$\frac{y}{x}=-\frac{2}{3}, \text{ which is } y=-\frac{2x}{3}, \qquad \text{A,}$$

is next to be considered. The quotient of y and x is here negative, being $=-\frac{2}{3}$: therefore they cannot have like signs, by [3]. If x is negative, as at q or n, y must be positive, and in length equal to $\frac{2}{3}$ of x, by A. Let

$$\left.\begin{matrix} Oq = \mathrm{Oq} \\ On = \mathrm{On} \end{matrix}\right\} \text{ then will } \left\{\begin{matrix} pq = \mathrm{pq} \\ mn = \mathrm{mn}, \end{matrix}\right.$$

for by a, $\mathrm{pq} = \frac{2}{3} \cdot \mathrm{Oq}$ or $= \frac{2}{3} \cdot Oq$; which by A is $= pq$, and

.... $\mathrm{mn} = \frac{2}{3}$ On or $= \frac{2}{3}$ On; .... $= mn$.

Now as the angle $X'OY =$ the angle XOY, the points p and m, which are found by the same measurements along the axes with the points p and m, form with O the same figure that p and m form with O, the figure on the right being exactly what the figure on the left is when seen *through the paper.* But O, p, m, have been proved to be in a line; therefore O, p, m, are in a line, and in the same way it can be shown that every point in the locus A is in this line Opm, which extends in opposite directions from O, as Opm does from O.

We have thus demonstrated that $y = \frac{2x}{3}$, and $y = -\frac{2x}{3}$ are true of points (xy) that lie in given right lines, and of no other points. These lines pass through the origin, and these equations are called the *equations to those lines.* If for $\frac{2}{3}$ any other number be substituted in either of the equations, the ordinate y corresponding to *a given value of* x, as $x = 1$, will be lengthened or shortened, and a different line through the origin will be represented for every different multiplier of x. If for $\frac{2}{3}$ be substituted, in all the preceding argument, the symbol e, *representing the fraction* $\frac{2}{3}$, then will it be proved equally, that $y = ex$, and $y = -ex$, are equations of the same lines through the origin: and if e represents *any constant number, which does not vary with* x *and* y, then it is equally proved, that $y = ex$ is the equation to the *corresponding line* through the origin, and this whether e have a positive or a negative value. We have thus established the following proposition:

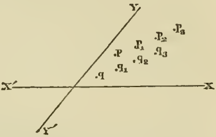
The locus of the points (xy) *whose co-ordinates satisfy the*

equation y = ex, e *being any constant number, is a straight line through the origin.*

You may say this briefly thus:

[4] ēx is ý gives óri. lí. ēx long.

7. We find a point in the locus $y = ex$ by taking in OX a length x_1 for x, and raising at the extremity of x_1 an ordinate of e times that length, ending at a point q. If we carried this ordinate one inch further in the positive direction beyond q, we should arrive at a point p, in the locus $y = ex + 1$. By adding an inch to the ordinate of any other point q_1 q_2 q_3 &c., found in the locus $y = ex$, we should obtain as many points p p_1 p_2 p_3 &c. in the locus $y = ex + 1$. The points p_1 p_2 p_3, &c., would form a line everywhere an inch distant along the ordinate from the line q q_1 q_2 q_3, and would therefore lie in a parallel to the line $y = ex$; for these lines can evidently never meet.



If we add a negative instead of a positive inch to every ordinate, i. e. retreat an inch along the ordinate from q q_1 &c. in the direction OY_1, we shall obtain a series of points similarly placed on the negative side of the line q q_1 q_2 ..., being all points in the locus $y = ex - 1$. If $qp = q_1p_1 = q_2p_2 =$ &c. had been b inches instead of 1, b being any number of either sign, whole or fractional, p p_1, &c. would have been points in the locus $y = ex + b$. We shall often speak of this locus, as the line $y = ex + b$. We have thus established

The locus of the points whose co-ordinates satisfy the equation y = ex + b, *where* e *and* b *are constants, is a line parallel to the line through the origin whose equation is* y = ex.

$y = ex + b$ gives, by subtracting b from both sides, $y - b = ex$, and this is evidently the same locus with $y = ex + b$. Put Dif. for *dif*ference: call $y - b$ Dif. (ȳb), (pron. wȳb), the difference between y and b, or y minus b: let parl. stand for parallel. Then

[5] The line (Dif. ȳb is ēx) is párl. to (y is ēx).

8. *Every line through the origin has an equation of the form* y = ex. For let $(x = x_1, y = y_1)$ be a point in a given

line, then is $y=\frac{y_1 x}{x_1}$ a line through the origin, by [4], for $\frac{y_1}{x_1}$ is a constant number; and this locus contains the points (0, 0), and (x_1, y_1), two points in the given line, as appears, if for x and y in the equation be put their values at those points; wherefore this is the equation to none other than the given line. Here x_1 and y_1 are two known numbers.

Every line has an equation of the form $y = ex + b$. For every line is parallel to *some line* through the origin. Let a given line be parallel to the line $y=e_1 x$, and let it meet the axis of y at a distance b_1 from the origin; e_1 and b_1 being known numbers, then is the equation $y=e_1 x+b_1$ that of *a* line parallel to $y=e_1 x_1$, [5], and of one which passes through the point $(x=0, y=b_1)$. Through this point there can only be one line, viz. the given one, drawn parallel to $y=e_1 x$; therefore $y=e_1 x+b_1$ is the equation to the given line.

9. *Every line has an equation of the form* $\frac{x}{l}+\frac{y}{a}=1$.

Read x by $l+y$ by a equals 1.

Let a given line be, e and b being given positive numbers.

(c) $$y=ex+b;$$ then, dividing equals by b,

$$\frac{y}{b}=\frac{e}{b}\cdot x+1,$$ or subtracting from both $\frac{e}{b}x$.

$$-\frac{e}{b}x+\frac{y}{b}=1,$$

(c′) $$-\frac{x}{\frac{b}{e}}+\frac{y}{b}=1;$$

for the fraction $\frac{ex}{b}$ is unchanged in value by the dividing both the numerator and denominator by the same number e. Thus (c) is reduced in (c′) *to the form* $\frac{x}{l}+\frac{y}{a}=1$, in which l and a, being general symbols, may of course have any particular values $a=b$, and $l=-\frac{b}{e}$. The equation under this form (c′) exhibits visibly the position and course of the line (c): for it shows that if $x=0$, $y=b$; and that

if $y = 0$, $x = -\frac{b}{e}$; i.e. the line meets OY at b positive inches from the origin, in the point $(0, b)$, and OX at $\frac{b}{e}$ negative inches from the origin in the point $(-\frac{b}{e}, 0)$. Thus we obtain instantly from (c′) two points of the line, and can therefore draw it, when our axes are given: and these *are always supposed to be given.* We can thus draw, for example, the line

$$y = \frac{3}{4}x + \frac{2}{7}; \quad \text{which is, dividing the equals by } \frac{2}{7},$$

$$\frac{y}{\frac{2}{7}} = \frac{3x}{4 \times \frac{2}{7}} + 1,$$

$$\frac{y}{\frac{2}{7}} = \frac{x}{\frac{4}{3} \cdot \frac{2}{7}} + 1 \quad \text{or, subtracting from the equals } \frac{x}{\frac{8}{21}},$$

$$\frac{y}{\frac{2}{7}} - \frac{x}{\frac{8}{21}} = 1;$$

this is now *of the form* $y{:}a + x{:}l = 1$, a being the fraction $2{:}7$, and l being $-8{:}21$. The former is the length cut off from O on the axis of y; the latter is that intercepted from the origin on the axis of x.

It will often be convenient to write the fractions $\frac{2}{7}$, $\frac{y}{a}$, &c. in the form $2{:}7$, $y{:}a$; and you may read these either two to 7, y to a, or two by seven, y by a; the *ratio* of 2 to 7 is the same number as the *quotient* of 2 by 7.

To remember this, you may add to the last mnemonic [5], the words

[5′] One 's 'vi (xl) and vi. (yà) pron. *vixle: and* is +.
From 'Or. cuts l and à.

Here *vi.* denotes *quotient of:* division; One 's means *One is*, or *One* =. The line 1 = quote of (y by l) and quote of (x by a) *cuts from* Origin (on OX and OY) the intercepts *l and a.*

The constants b, e, l, a, in [5] are not in general whole numbers; nor would every line be reducible to the forms $y = ex + b$, and $x{:}l + y{:}a = 1$, if these four constants were

not capable of representing *any* numbers of either sign. If a line is to be represented in general without fractions, three constants are required in its equation.

10. *Every line has an equation of the form* $\mathrm{A}x + \mathrm{B}y = \mathrm{C}$, *in which the constants are not fractions.*

Thus the line $y = \frac{3}{4}x + \frac{2}{7}$, by multiplying both sides by 4×7,

becomes $28y = 21x + 8$, or, taking $21x$ from both equals,

$$-21x + 28y = 8,$$

which is *of the form* $Ax + By = C$, without fractions.

Show now that the line $13x + 5y = 8$ is parallel to $x{:}y = -13{:}5$, and that it cuts off from the origin $a = 8{:}5$ and $l = 8{:}13$.

To draw any given line $y = ex$, it is only necessary to find one point of it, and to join that to the origin. The point $(1, e)$ is such a point, evidently.

11. *The axis of* y *is* $\mathrm{x} = 0$; *the axis of* x *is* $\mathrm{y} = 0$. No point whose x is not nothing is on the former axis; nor is any whose y is not nothing on the latter. Do you see this, Richard?

Richard:—Plainly: and it is evident that every point whose x is nothing is in OY, and that every point whose ordinate is nothing is in OX.

Jane:—But I do not see these equations, $y = 0$ and $x = 0$ are of the form $y = ex$: if e have the value zero, indeed, $y = 0$ is the result; but what must e be to bring out $x = 0$?

Uncle Pen.:—The equation $y = ex$ is true still when both sides are divided by e: i. e. $\frac{y}{e} = x$, or $y \times \frac{1}{e} = x$, follows from $y = ex$. Now the fraction $\frac{1}{e}$ diminishes as e increases: if e increases beyond all conceivable limit, $\frac{1}{e}$ lessens beyond all limit; if e becomes infinite, $\frac{1}{e}$ dwindles to zero, and the equation $y \cdot \frac{1}{e} = x$ is then $0 = x$. The equation $y = 0$ is the case of $y = ex$, which arises from the supposition, $e = 0$: $x = 0$

is that which springs from the supposition, $e=\frac{1}{0}$, which is the symbol of an infinite number. Let r be an indefinitely small number; the quotient $\frac{1}{r}$ is indefinitely great: e.g. $\frac{1}{0{\cdot}0000001}$ is ten millions. If $r=({\cdot}0000001)^2$, and $e=\frac{1}{r}$, e is 100 billions: if in r the number of zeros between the point and unity be unlimited, $r=0$, and $e=\frac{1}{r}$ is then $e=\frac{1}{0}$, which is greater than any finite number. You know that the number $\frac{1}{5}$ is called the reciprocal of 5. Infinite is the reciprocal of zero.

Jane :—I know also that a number *times* its reciprocal is equal to unity; $5\times\frac{1}{5}=1$. Is then $0\times\frac{1}{0}=1$?

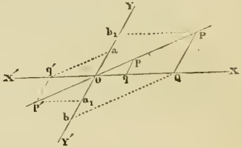
Uncle Pen.:—The quantity $0\times\frac{1}{0}$ or $\frac{0}{0}$ is called an indeterminate quantity. Zero by zero gives any quotient you please; for 5×0, and 1×0, are equally 0. This is sufficient for you to know *at present:* we shall not have any occasion for some time to handle the quantity $\frac{0}{0}$.

Consider the two lines $y=ex+b$, $y=-ex+b$, e and b being the same pair of fixed numbers in both. Both equations are true at $(0, b)$ i.e. the two lines meet in $(0, b)$. When $x=n$, the same value in both, we obtain $y_1=en+b$, $y_2=-en+b$, the subindices serving for distinction between the two ordinates, which are both measured on the same parallel to OY. By addition, $y_1+y_2=2b$, whatever be the values of e and n. Draw a parallel to OX through $(0, b)$: we have proved that if this and our two lines be cut in A, B, C, by a parallel to OY through *any* point P of OX, $PB+PC=2PA$. Draw the figure and examine this.

LESSON II.

Richard:—When shall we be able to understand the secret that you promised to show us, how to measure the height of a tower without ascending it?

Uncle Pen.:—You shall know it presently, if you will pay close attention to the following argument.



12. Let OP be a line through the origin and any given point P. If we put the co-ordinates PQ, OQ, &c. for their lengths in inches,

$$\left.\begin{aligned}&\text{we know that } y = \frac{PQ}{OQ}x,\\ &\text{or, dividing by } x, \frac{y}{x} = \frac{PQ}{OQ},\end{aligned}\right\}\ldots\ldots(A)$$

is the equation to OP; for by [4], this is *a* line through the origin O, and it is true for ($y = PQ$, $x = OQ$), so that the line whose equation is (A) contains *two points of OP*. If p and p' are points in OP,

$$\text{B.}\qquad \frac{pq}{Oq} = \frac{PQ}{OQ}, \quad\text{and}\quad \frac{-p'q'}{-Oq'} = \frac{PQ}{OQ}, \qquad \text{C.};$$

for (A) must be true at both these points. You may read B either pq by Oq equals PQ by OQ, or pq to Oq as PQ to OQ, and I recommend you *to read every equation aloud.*

If we choose now to consider OP and OY, instead of OX and OY, to be our positive axes of x and y, the point Q referred to these, is

$$(x' = OP,\ y' = -Ob \text{ or } -QP) \quad [2].$$

And the equation to OQ is

$$\frac{y'}{x'} = \frac{-PQ}{OP} \qquad (A').$$

for (A′) is *a* line which contains *two* points of OQ, and can

therefore be none other than OQ. The dashes merely show that y' and x' are co-ordinates referred to the new axes.

As (A′) must be true at q and q', points of OQ,

b. $$\frac{-qp}{Op} = \frac{-PQ}{OP}, \quad \text{and} \quad \frac{q'p'}{-Op'} = \frac{-PQ}{OP},$$ c.

From the pair of equations

B. $$\frac{pq}{Oq} = \frac{PQ}{OQ},$$

b. $$\frac{pq}{Op} = \frac{PQ}{OP},$$

comes, multiplying the equal sides of B by the fraction $Oq{:}PQ$, and the sides of b by $Op{:}PQ$, the pair

B′. $$\frac{pq}{PQ} = \frac{Oq}{OQ}, \quad \text{and} \quad \frac{pq}{PQ} = \frac{Op}{OP},$$ b′.

And by taking the quotient of the left members of B and b, and that of their equal members on the right, we get

Bb. $$\frac{Op}{Oq} = \frac{OP}{OQ}.$$

In B′, $\frac{pq}{PQ} = \frac{pq}{Oq} \cdot \frac{Oq}{PQ}$, &c. In Bb, $\frac{Op}{Oq} = \frac{pq}{Oq} \div \frac{pq}{Op} = \frac{pq \cdot Op}{Oq \cdot pq}$.

From the pair of equations,

C. $$\frac{p'q'}{Oq'} = \frac{PQ}{OQ},$$

c. $$\frac{p'q'}{Op'} = \frac{PQ}{OP},$$

we obtain in the same manner, with $q'p'$ for qp, for B' and Bb.

C′. $$\frac{p'q'}{PQ} = \frac{Oq'}{OQ}, \quad \frac{p'q'}{PQ} = \frac{Op'}{OP},$$ c′.

Cc. $$\frac{Op'}{Oq'} = \frac{OP}{OQ}.$$

I have omitted the negative signs from b; because from such a truth as $\frac{-3}{2} = \frac{-6}{4}$, comes $\frac{3}{2} = \frac{6}{4}$, [3], multiplying both sides by -1. And I have left out the signs in the left

member of C; for $\frac{-3}{-2}=\frac{3}{2}$, [3]; for $\frac{q'p'}{-Oq'}$ in c I put its equal $\frac{-q'p'}{Oq'}$, and then omit negative signs as in b.

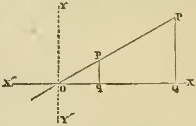
Do you understand all this?

Jane:—I think I see that every step of the argument is proved; but I know not where I am, or what is before me, and cannot see much of what is behind me. It is like plunging into a dark cavern guided by a slender thread: I have just hold of it, and that is all.

Uncle Pen.:—It will never break, for the twine is indestructible; and there will be light enough presently. If you are convinced that equals divided by equals give equal quotients, you are certain that Bb and Cc are true; and if equals multiplied by the same quantity remain still equals, B′, b′, C′, c′, are true likewise. When you see all the meaning and application of these results, you will know that they contain the whole science of geometry. If you multiply both sides of B by OQ, you get

$$PQ=\frac{pq}{Oq}\cdot OQ, \quad \text{read } (pq \text{ by } Oq) \text{ times } OQ,$$

which contains the secret of the tower. Suppose the axes chosen rectangular, for YOX, as well as POX, may be any angle we please; and if the co-ordinates are parallel to the axes, all our equations remain unaltered in their truth.



Let PQ be the perpendicular tower; let p be any point which your eye at O sees in a line with the summit P; e.g. Op may be a telescope directed to P, p being the centre of the object-glass. If you know the length of OQ, the horizontal distance of your eye at O from the tower; and know too the length of the perpendicular Pq, which is parallel to PQ and to the axis of y, and also the length of Oq, you obtain PQ from the last written equation, by multiplying together the lengths pq and OQ, and dividing the product by the length Oq. Adding to this quotient the height of your eye from the ground, you have the number of inches or feet between P and the ground, (supposed there on a level with your foot), according as your other lengths are expressed in inches or in feet. Equation *Bb*, which is

$$OP = \frac{Op}{Oq} OQ, \qquad \text{D.}$$

gives the distance OP, if you first know the lengths Op, Oq, and OQ. You will shortly learn how to determine either PQ or OP, when only OQ and the magnitude of the angle POQ are known.

Richard:—How very charming! All this comes out of our first notion of the point, $(x=3, y=2)$, and a little division and multiplication, thus,

$$y=2,$$

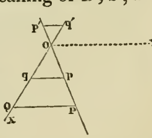
$$x=3,$$

$$\text{giving, } \frac{y}{x} = \frac{2}{3}, \text{ and then, } y = \frac{2x}{3};$$

this is in fact the height of the tower, if x is OQ, and p happens to be the point (3, 2). I do not despair now of measuring mountains in the Moon, a feat that you were helping cousin Henry last week to perform.

Jane:—I see that you are right, if the axes are rectangular. How simple, after all! The two first equations written by Richard are true of no x and y but those of the point (3, 2); the next expresses a law by which x and y may vary through all the points in the line Op. How delighted y must feel, in the third of these equations, to be free from his confinement in the first, and to be able to assume any value, positive or negative, that pleases him, compelling x every moment to assume the corresponding value!

13. *Uncle Pen.*:—To see now the meaning of B′, b′, and Bb, look at the figure thus: OP and OQ are any pair of diverging (legs or) lines, which are cut by the parallels PQ and pq, drawn in any direction (OY). Op, OP are the segments of one leg, and Oq, OQ those of the other, made by the parallels. From B′ and b′ we learn that



the ratio or proportion of Op to OP, (i.e. their quotient) is the same number as the ratio of Oq to OQ: and that either of these ratios is the same with that of pq to PQ. From Bb we see that the ratio of the two segments made by pq is equal to that of the two made by PQ. In this case the two cutting parallels are on the same side of O. If pq moves

parallel to itself to any position on the farther side of O from PQ, as to $p'q'$, we see from C′, c′, and Cc, that the same things are still true, putting p' for p and q' for q. Thus is proved that

If a pair of parallel lines (PQ, pq) *cut a pair of lines meeting in a point* (O), *the ratio of the two segments* (OP, OQ) *made by one cutting line is equal to that of the corresponding segments* (Op, Oq) *made by the other: and the ratio of the intercepted parallels* (Pq, pq) *is equal to that of the corresponding segments cut off by them in either line,* (OQ, Oq) *or* (OP, Op); *the segments being measured all from* O.

This is easily remembered in a condensed shape thus:

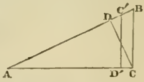
[6] If páralls. cut légs,
vi. ségs. is vi. ségs.,
And ví. (parall. cútters) is ví. (corre. ségs.):
Measure the segs. from meet. of legs.

Here vi. means *quote of*, as in [5′]. Segs. stands for segments. *Corre.* is *Corre*sponding; meet. for meeting.

You are *not* to lay down $PQ : pq$ as $Op : OP$. I advise you to learn the mnemonic *first*, and to meditate afterwards with this *at your tongue's end;* for ready words are instruments of ready thought.

14. You can now easily understand the proof of that most renowned theorem of Pythagoras, (*Euclid*, I. 47) on the discovery of which he is said to have sacrificed a hecatomb.

Let ABC be any triangle right-angled at C: let CD be a perpendicular from C on AB, the *hypothenuse*, or side *opposite* the *right* angle, meeting that side in D. Cut off on AB, $AC' = AC$, and on AC, $AD' = AD$. Then joining $C'D'$, we see plainly that the triangle $AD'C'$ is the triangle ADC turned face downwards, so that $AD'C'$ is a right angle like ADC and ACB, wherefore $D'C'$ is parallel to BC, both being at right angles to the same line AC. Because the parallels BC and $D'C'$ cut the legs AC and AB [6],



$$\frac{AC}{AD'} = \frac{AB}{AC'},$$

or, since $AD = AD'$ and $AC = AC'$,

$$\frac{AC}{AD} = \frac{AB}{AC};$$

whence, multiplying these equals by $AC \cdot AD$, (read AC times AD),

(a) $$AC \,.\, AC = AB \,.\, AD.$$

We have here proved, that if in *any* right-angled triangle D is the point in which the perpendicular from the right angle C meets the hypothenuse, the squared length of the base AC is the product of the lengths of the hypothenuse AB and segment AD adjacent to the base. This will therefore be true of *our* triangle when made to stand on the base BC, or

(b) $$BC \,.\, BC = BA \,.\, BD.$$

Hence, by addition of the quantities on the left of (a) and (b) and of their equals on the right,

$$AC \,.\, AC + BC \,.\, BC = AB \,.\, AD + AB \,.\, DB$$
$$\text{or} = AB \,.\, (AD + DB) = AB \,.\, AB.$$

i. e. $$(AC)^2 + (BC)^2 = (AB)^2, \qquad \text{(c)}$$

or *the square of the length of the hypothenuse* AB *is the sum of the squares of the lengths of the sides* AC *and* BC. Hence if the sides about the right angle are 3 and 4, the hypothenuse must be *5*; for $5^2 = 3^2 + 4^2$: 6 and 8 for the sides would give 10 for the hypothenuse. Every bricklayer knows that 6, 8, and 10, will make a right-angled triangle. Suppose the two sides were each $= 1$; then $1^2 + 1^2 = 2$, shows that $(AB)^2 = 2$, or $AB = \sqrt{2}$. Thus you see there are lines which cannot be measured; for no fractional number can be found that expresses the number of inches in the diagonal of a square whose side is one inch. By extracting the square root of 2 you find for AB that diagonal 1·41421356237 inches, which is correct to the ten thousand millionth part of an inch, but not *quite* correct; in truth, the decimal has no end. The side of a square and its diagonal are called *incommensurable* quantities: no scale can measure both; no number can express what part one is of the other.

Richard:—But suppose that two divisions of my scale happened to be exactly the length of the diagonal of a square: surely 2 would then express the length of it?

Uncle Pen.:—It would: but you would not be able to measure, or even exactly to calculate the side. If AC and BC are supposed equal, and $AB = 2$, as you propose; our equation would give $(AC)^2 + (AC)^2 = 2^2$, or $2\,(AC)^2 = 4$, or $(AC)^2 = 2$, or $AC = \sqrt{2}$, a *number incommensurable* with unity. But although we cannot measure or write out correctly

such a quantity as $\sqrt{2}$, it is nevertheless a *number;* and we can write down a symbol for it, and reason correctly about it. Thus $\sqrt{2}$ is such a *symbol,* and if we say, let $AC=\sqrt{2}$, or let $PQ=\sqrt{2}$; AC or PQ is such a symbol, and we can reason with it, as accurately as with an integer number.

By our suppositions thus far, our symbols (AC), (AB), &c. stand for numbers of inches in certain lines: and the conclusion $(AC)^2+(BC)^2=(AB)^2$ is an assertion about numbers only. Let us repeat our argument about the square of the hypothenuse, putting all through for AC $AC.I$, for BC $BC.I$ &c., and let I denote a linear visible inch; then $AC.I$ denotes AC times such a visible line, and becomes a line, although AC is but an abstract number. The conclusion (c) above, will be modified thus,

$$AC.I.AC.I+BC.I.BC.I=AB.I.AB.I$$

$$\text{being } =A.B.I\,(AD.I+DB.I),$$

$$\text{or} \qquad (AC.I)^2+(BC.I)^2=(AB.I)^2 \qquad \text{(c}')$$

Since I is no number, but a line, I^2 is no number, but a unit line multiplied by itself; and in like manner $(AC.I)^2$ is a line multiplied by itself. This has no obvious meaning; it must be defined; and it is in our power to give it any definition consistent with the operations of our arithmetic. *We shall define a line multiplied by another line, to be a rectangle, or right-angled parallelogram, whose adjacent sides are the two lines:* then I^2 is a square inch, and $(AC.I)^2$ is the square upon AC inches. The last equation is now an assertion about square spaces, and affirms: that the square upon the line AB is an area equal to the two squares on AC and BC. There is nothing to hinder us from interpreting equation (c) at once to mean the same thing without introducing the linear unit I, if we bear in mind the definition just given of the product of two lines. In future, when we are considering certain points A, B, C, D..., we shall take the symbol AB to represent either the *line* drawn from A to B, or the *number* of inches in that line. *Arithmetically* viewed, the line and the number are the same. If A represented a *number,* instead of a *point,* and B were also a number, AB would mean A times B, a product of two numbers; but we know always whether a symbol A or B stands for a point or a number, and there can here be no confusion. When $ABCD$ are points, $AB.AD$ may be either the product of the two

numbers AB and AD, or the *right-angled space* AB inches by AD inches, according as we consider AB and AD to be numbers, or visible lines; and the same may be remarked of $AB \, . \, CD$.

The square upon the hypothenuse of a right-angled triangle is equal to the sum of the two squares upon the sides.

This is the famous Theorem of Pythagoras. By the square upon a line is meant the square whose side is that line: remember that there is no hypothenuse, where there is no right angle.

If we put qua. for *qua*drate or sq*ua*red; and *poth.* for hy*poth*enuse, this proposition may be fastened to the ear and to the tongue in the condensed form following:

[7] qua. póth. is bóth qua. sídes.
sq*ua*red hy*poth*enuse is = *both* the sq*ua*red *sides.*

15. By (a) above

$$(AC)^2 = AD \, . \, AB \qquad \text{(a)}$$

and $$(AC)^2 = (AD)^2 + (DC)^2 \qquad \text{by [7]},$$

since the triangle ADC is right-angled at D. Wherefore

$$(DC)^2 + (AD)^2 = AD \, . \, AB, \qquad \text{whence,}$$

subtracting $(AD)^2$ from both sides, leaving equal remainders,

$$(DC)^2 = AD \, . \, AB - (AD)^2 = AD \, . \, (AB - AD), \text{ or}$$

$$(DC)^2 = AD \, . \, DB. \qquad \text{(d)}$$

If you multiply together two numbers and then find the square root of the product, this root is called the mean proportional between the numbers: thus 6 is the mean proportional between 4 and 9, because $\sqrt{4 \times 9} = 6$, or $6^2 = 4 \times 9$. This root happens to be commensurable (with unity); but the mean proportional between 5 and 9 is not, and can only be expressed by the *symbol* $\sqrt{5 \times 9}$, or $\sqrt{45}$. These are called *mean* proportionals, because of the proportions $4 : 6 :: 6 : 9$, and $5 : \sqrt{45} :: \sqrt{45} : 9$; both which are true by the Rule of Three.

As (d), (a), and (b), give

$$DC = \sqrt{AD \, . \, DB}, \quad AC = \sqrt{AD \, . \, AB}, \text{ and } BC = \sqrt{BD \, . \, AB},$$

we see that the perpendicular CD is the mean proportional between the two segments AD and BD, which it makes of the hypothenuse; and either side, AC, or BC, is the mean

proportional between the hypothenuse, and that one of these two segments, which is adjacent to itself. For what is true of the proportions of the lengths of lines in numbers is true of the proportions of the lines. This proves the following:

In any right-angled triangle, the perpendicular let fall from the right angle on the hypothenuse is the mean proportional between the segments into which it divides the hypothenuse; and either side of the triangle about the right angle is the mean proportional between the hypothenuse and that so-made segment thereof, which is adjacent to that side.

Let us put *mean* or *mea.* for (*mean* proportional between), and *per.* for *per*pendicular, and *poth.* as in [7]. We may say

pér. on póth. is méan segs;

[8] and síde is méa. (poth. nígh seg.)

side means *either side*; *nigh seg.* is the *segment adjacent.*

Ex. 1. OPQ is a triangular field whose sides are $OP = 1000$, $OQ = 840$, $QP = 380$ feet. Oq being 300 feet, what is the length of the line qp crossing the field parallel to PQ, and what is the distance Op?

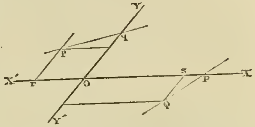
The solution is obtained from [6]. Equations B and Bb, page 17, give $pq = 135\frac{5}{7}$ feet, and $Op = 357\frac{1}{7}$.

Ex. 2. The perpendicular on the hypothenuse is p, and one segment of the same is s: what are the sides of the triangle?

By [8] and equation (d), the other segment is $p^2{:}s$, so that the hypothenuse is $s + p^2{:}s$. By [7] the sides are $\sqrt{s^2 + p^2}$ and $\sqrt{p^2 + (p^2{:}s)^2}$. If $p = 3$ and $s = 1$, the hypothenuse is 10, and the two sides are $\sqrt{90}$ and $\sqrt{10}$.

LESSON III.

16. I SHALL now propose and solve an entertaining problem. OX and OY are lines in the same plane in a dense and extensive forest. Two points pq mark one line, and two points PQ mark another, which are to be central lines of two level roads to be made in the same plane in the forest. A third road is to be formed in this plane also, parallel to the line OY, from the intersection of the roads pq, PQ.



The engineer requires to know where the third road will cross the line OX, and how far the crossing point is from the intersection of the other two roads pq and PQ.

We can answer this by finding the point in which the line pq meets PQ; the co-ordinates of this point, if our axes are OX and OY, are the lengths required. This point can be found, when we know the equations of the lines pq and PQ. Our first step must be

To find the equation of a line from two given points in it.

The equation sought is *of the form* (10)

$$Ax + By = C, \ldots\ldots 1,$$

and if $(x_1 y_1)(x_2 y_2)$ be the given points, this must be true at both, whatever numbers A, B, and C may be; or

$$Ax_1 + By_1 = C \ldots\ldots 2.$$
$$Ax_2 + By_2 = C \ldots\ldots 3.$$

If we subtract the left member of 2 from that of 1, and do the same with their right members, i. e. take equals from equals, the remainders are equal, or

$$Ax + By - Ax_1 - By_1 = 0, \text{ or}$$
$$A.(x - x_1) + B.(y - y_1) = 0,$$

where the *point* is to be read, *times*, or

subtracting $B.(y - y_1)$ from these equals,

$$A.(x - x_1) = - B.(y - y_1) \ldots\ldots 4.$$

B

Repeating both these steps with equations 2 and 3, instead of 1 and 2, we obtain

$$A\,.\,(x_1 - x_2) = -\,B\,.\,(y_1 - y_2) \,\ldots\ldots\ldots\ldots\, 5.$$

Dividing the equal members of 4 by the equals in 5,

$$\frac{A\,.\,(x - x_1)}{A\,.\,(x_1 - x_2)} = \frac{B\,.\,(y - y_1)}{B\,.\,(y_1 - y_2)},\ \text{by [3]};$$

for in 4 and 5, the quantities on the right have *like* signs; therefore their quotient is *not* negative: this is evidently

$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2} \,\ldots\ldots\ldots\ldots\ldots\, 6.$$

Multiplying these equals by $(y_1 - y_2)$,

$$\frac{(y_1 - y_2)\,.\,(x - x_1)}{x_1 - x_2} = y - y_1;\ \text{then by}\ (x_1 - x_2),$$

$$(y_1 - y_2)\,.\,(x - x_1) = (x_1 - x_2)\,.\,(y - y_1) \,\ldots\ldots\ldots\, 7.$$

Or 6 gives 7 at one step, if you multiply by $(x_1 - x_2)\,.\,(y_1 - y_2)$.

The assertion $(7-3) \times (4-2) = (9-5)\,.\,(8-6)$, amounts to

$$7 \times 4 - 7 \times 2 - 3 \times 4 + 3 \times 2 = 9 \times 8 - 9 \times 6 - 5 \times 8 + 5 \times 6; \quad [3].$$

and 7 gives

$$y_1 x - y_1 x_1 - y_2 x + y_2 x_1 = x_1 y - x_1 y_1 - x_2 y + x_2 y_1,\ \text{whence}$$

$$x y_1 - x y_2 + x_1 y_2 = x_1 y - x_2 y + x_2 y_1, \qquad \text{or adding } x_2 y \text{ to both sides,}$$

$$x y_1 - x y_2 + x_2 y + x_1 y_2 = x_1 y + x_2 y_1,$$

and subtracting $x_1 y + x_2 y_1$ from both,

$$x y_1 - x y_2 + x_2 y + x_1 y_2 - x_1 y - x_2 y_1 = 0,\ \text{which is}$$

$$\text{either}\ x\,.\,(y_1 - y_2) + x_1 (y_2 - y) + x_2 (y - y_1) = 0 \,\ldots\ldots\, 8,\ \text{or}$$

$$(y_1 - y_2)\,.\,x + y\,.\,(x_2 - x_1) - x_2 y_1 + x_1 y_2 = 0,\ \text{or}$$

$$(y_1 - y_2)\,.\,x + (x_2 - x_1)\,.\,y = x_1 y_2 - x_1 y_2, \,\ldots\ldots\ldots\, 9,$$

which has the form, $Ax + By = C \ldots\ldots 1$.

The equations 6, 7, 8, 9, are all forms of the same equation to the line required. *We can see with our eyes,* in 6, that the line passes through the points $(x_1 y_1)$ and $(x_2 y_2)$. If we multiply the equal sides both by -1, which merely changes the signs, for

$$(x - x_1) \times -1 = -x + x_1 = x_1 - x,$$

$$\text{we have}\ \frac{y_1 - y_0}{y_1 - y_2} = \frac{x_1 - x_0}{x_1 - x_2} \ldots\ldots (6.)$$

Here x_0, y_0, still mean the variables xy, the zeros being appended for the sake of symmetry and memory. If you put now x_1 and y_1 for x and y, (6) becomes $0 = 0$; if you put x_2 and y_2 for them, it becomes $1 = 1$. The equation is thus true, *it is satisfied*, at both these points; therefore the line represented by it passes through them. It was true also as it stood before; giving $0 = 0$, and $-1 = -1$.

Equation 8 is best remembered thus, putting $x_0 y_0$ for the variables,

$$x_0 . (y_1 - y_2) + x_1 . (y_2 - y_0) + x_2 . (y_0 - y_1) = 0 \ldots\ldots (8).$$

17. It is of importance to remember the form of 6; for having this, you have all the following forms by easy transformation, which will soon become familiar. The forms 6 and 8 are easily retained from their symmetry. We are informed by (6), that the quotient of two differences of y's = the quotient of two differences of x's, the subindices being on both sides, 10 above and 12 below. In 8 we see that x times a difference of y's, thrice written, = 0; the subindices being in order 012, 120, 201. These three terms are made by writing 012, and then carrying the first figure to the last place as often as possible, 012, 120, 201, 012, 120, &c. You cannot *thus* obtain more than three arrangements, by going *round the circle*, so to speak. To remember then these forms say, the quotient of differences of y's is the quotient of differences of x's, with 10 over 12 for the subindices. This is (6). And for (8) say: *Thrice* written x *d*ifference of y's = 0; thus,

$$x . (y - y) + x . (y - y) + x . (y - y) = 0;$$

then go *round* the circle 012 for the subindices, 012, 120, 201.

Let *vi* (*divi*de) stand for quotient as in [5]′: let *vi.D*, mean quote of *D*ifferences. You may thus abbreviate (6), putting *–dex* for subin*dex*, & *is* for =,

[9] vĭD(y's) is vĭD(x's), read *viD-wise.*
with 10 o'er 12—déxes: read ten o'er twelve.

and thus 8; ter = thrice; Di = *Di*fference; *is* for =; nil = 0;

tĕr x′Dĭ(y's) is níl,
at ó12 roúnd, for gíl. *gil* is *g*iven *l*ine.

pronounce owe un two, vowel *o* for zero, *un* for *unity*.

18. We have now *to find the intersection of two given lines.*

They *must* be of the *forms*, (10),

$$Ax + By = C, \qquad (A)$$
$$A'x + B'y = C'; \qquad (B)$$

and these will represent *any* pair of lines, when the known and proper values are put for the six constants. As equals multiplied by any number are still equals,

$$A' . (Ax + By) = A'C,$$
$$A . (A'x + B'y) = AC'.$$

Read these, A' times (the sum of Ax and By) equals $A'C$, &c. Either of these must be true of every (x, y) in the line represented: they are then *both true* of (X, Y) the point of intersection. Putting then X and Y for x and y, and performing the multiplications indicated in the left members,

$$A'AX + A'BY = A'C,$$

$$AA'X + AB'Y = AC'; \text{ whence by subtraction,}$$

$$AB'Y - A'BY = AC' - A'C,$$

which must be true, although we know not yet either X or Y; that is,

$$(AB' - A'B)\,Y = AC' - A'C, \quad \text{whence by division of equals,}$$

$$Y = \frac{C'A - CA'}{B'A - BA'}. \tag{C}$$

This gives us the value of Y the ordinate of the intersection, since all the numbers on the right are known: we have only to subtract C times A' from C' times A, and divide the remainder by the difference ($B'A$ minus BA'): the quotient is the value of Y. It remains to find X; and this will of course be given by either (A) or (B), if we put for y in either the number Y. But the symmetry of our equations will help us more expeditiously to the number X. It is clear that in (A) you may put x for y, if you exchange B for A; and in (B) likewise, if you exchange B' for A': this makes it highly probable that X may be put for Y in (C) if B and D' be exchanged for A and A'. The probability becomes a certainty if you write

$$Bx + Ay = C, \tag{A}$$

$$B'x + A'y = C', \tag{B}$$

and then repeat the above process with B for A, &c., which gives

$$X = \frac{C'B - CB'}{A'B - AB'}.$$

It is important that you should remember the above value for Y; and you may teach your ear the following rhyme:

[10] *C*'s *Ax* and *By*; *C*'s for *C* =.

Dot second li:

Dĭ(*CA*)'s bў Dĭ. (*BA*)'s, (dŏt oùts.), is mèeting *Y*.

You *dot* or accent your constants ABC, for your *second line*. *Diff*erence of (CA)'s by *Diff*erence of (BA)'s, *dot*ting the *out*side letters as in (C), *is* the Y of the *meeting* point. X is obtained from Y by putting A's for B's, and v. v.

The engineer's problem is now completely solved in general symbols; and all that remains is to put for ABC $A'B'C'$ their numerical values in the fractions that represent Y and X. These six constants are given in the equations to the lines pq and PQ; and these equations we can form instantly from [9] when the engineer has supplied us with the measurements that determine his four given points. Let p be (x_1y_1) and q be (x_2y_2); and let the measurements be given in miles, that is, let a mile be our unit of length. Suppose

$$x_1 = -2 = Or, \qquad x_2 = 0,$$

$$y_1 = 1\tfrac{3}{4} = pr, \qquad y_2 = 3 = Oq.$$ Then by [9]

$$x_0 \,.\, (y_1 - y_2) + x_1 \,.\, (y_2 - y_0) + x_2 \,.\, (y_0 - y_1) = 0, \qquad (8),$$

or substituting

$$x_0 \,.\, (1\tfrac{3}{4} - 3) - 2 \,.\, (3 - y_0) + 0\,(y_0 - y_1) = 0.$$

Here $x_1 = -2$, therefore $+x_1 = -2$, and $x_1 \,.\, (y_2 - y_0) = -2\,(y_2 - y_0)$.

$$\frac{-5}{4}x_0 + 2y_0 - 6 = 0\,; \qquad \text{for } -2 \times -y_0 = 2y_0,\ [3]:$$

hence $\frac{-5}{4}x_0 + 2y_0 = 6$, by adding 6 to the equal sides;

and $-5x_0 + 8y_0 = 24$, by multiplying both by 4 : (pq).

or $Ax + By = C$. (A).

Thus $-5x + 8y = 24$, or $\dfrac{x}{-\frac{24}{5}} + \dfrac{y}{3} = 1$, is the line (pq).

Next let Q be (x_1y_1) and P be (x_2y_2) in (8); and suppose

$$x_1 = 3{\cdot}25 = Os, \qquad x_2 = 3{\cdot}9 = OP,$$

$$y_1 = -1{\cdot}2 = Qs, \qquad y_2 = 0.$$

Substituting these values in (8) we obtain

$$x_0 \,.\, (-1{\cdot}2 - 0) + 3{\cdot}25\,(0 - y_0) + 3{\cdot}9 \,.\, (y_0 + 1{\cdot}2) = 0, \text{ or}$$

$$-1{\cdot}2x + (3{\cdot}9 - 3{\cdot}25)\,y + 4{\cdot}68 = 0, \text{ or}$$

$$-1{\cdot}2x + {\cdot}65y = -4{\cdot}68, \text{ or mul. by } -100,$$

$$120x - 65y = 468, \qquad (PQ)$$

$$A'x + B'y = C'. \qquad (B).$$

This, which is the same with $\frac{x}{3\cdot 9} + \frac{y}{-\frac{468}{65}} = 1$, is the line PQ.

$$\text{Our } Y \text{ [10] is } \frac{C'A - CA'}{B'A - BA'} = \frac{468 \times -5 - 24 \times 120}{-65 \times -5 - 8 \times 120}$$

$$= \frac{468 + 24 \times 24}{-65 + 8 \times 24} = \frac{1044}{127} = 8\tfrac{28}{127}.$$

This is the distance of the point of intersection of the roads pq and PQ from the line OP, along a parallel to OY, being 8 miles and $\frac{28}{127}$ of a mile. We find X from equation (pq) thus, by using this value of Y,

$$-5X + 8 \times \frac{1044}{127} = 24, \qquad \text{or, subtracting from both } 8 \times \frac{1044}{127},$$

$$-5X = 24 - 8 \times \frac{1044}{127},$$

$$-5X = \frac{3048 - 8352}{127} = -\frac{5304}{127}; \qquad \text{then dividing by } -5,$$

$$X = \frac{5304}{5 \times 127} = 8\tfrac{224}{635};$$

this is the distance in miles from O along OP to the centre of the road which is to cross OP. We could have found X also from equation (PQ).

19. *Jane:*—I have observed in the deduction of equations (PQ) and (pq), and of (8) and (9) just now, that *any quantity can be transposed from one side of an equation to the other, if care is taken to change the sign of the quantity transposed,* and that this transposition is always either an addition or a subtraction of that quantity, performed in both members of the equation.

Uncle Pen.:—Your observation will be *quite* correct, if you say *any term* instead of *any quantity:* a quantity so transposed must carry with it to the other side its multiplier or its divisor, if it has either. For the difference between a term and a quantity, look at equation (7); it has one *term* on each side, each term being a product of two factors, every factor being composed of two quantities tied by a *vinculum.* When the vincula are untied, which requires the indicated multiplications to be performed, the equation

has four terms on each side, as you see in the step following (7). In (8) you see three terms on the left; if you untie and multiply, you will have six terms in the left member of the equation.

Let me now see, Richard, whether you can find the value of X, corresponding to *meeting* Y of [10], by putting for y in equation (A) the expression for Y. First transpose the term By; then put Y for y, and next bring the quantities on the right to a common denominator.

Richard:—I have done this, and I cannot succeed.

I get $Ax = \dfrac{(CB'A - CBA') - (BC'A - BCA')}{B'A - AB'}$, which means

$$Ax = \frac{CB'A - CBA' - BC'A - BCA'}{B'A - AB'};\quad \text{then dividing both by } A$$

$$x = \frac{CB'A - CBA' - BC'A - BCA'}{AB'A - AAB'}.$$

This is not the value X that you found; where is the blunder?

Uncle Pen.:—You are correct in the first of these three equations; the second is *not* what the first *means*. You are right in saying

$(CB'A - CBA') = CB'A - CBA'$; and wrong in saying

$-(BC'A - BCA') = - BC'A - BCA'$;

for this lower term on the left, when *untied*, must have a sign contrary to that of $+(BC'A - BCA')$ or to that of $BC'A - BCA'$; i.e. it must be $-BC'A + BCA'$. Make the correction, and your numerator will then be divisible by A, and x after the division will shew its true value. Consider this:

$$5 + (2 - 5) - (7 - 3 + 2 + 1 - 6) = 5 + 2 - 5 - 7 + 3 - 2 - 1 + 6 = 1.$$

The first + on the left is the sign, not of 2, but of the tied or the vinculated quantity $(+2 - 5)$; the second − is the sign not of 7, but of $(+7 - 3 + 2 + 1 - 6)$. When these terms are *untied*, the signs of the ties or vincula disappear: and the rule is; *when you untie a term preceded by the sign −, you must change the sign of every quantity in that term.*

As you are a careless boy, Richard, I shall require you to repeat the following bad rhyme:

[11] After minus untyin'
Change every sign.

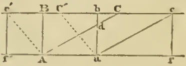
Jane:—That was a pretty device, by which you got rid of x from the equations (A) and (B), (18); by a pair of multiplications and a subtraction—so short and simple!—and then a division equally simple compelled the y to show himself. How surprised he must have been, after see-sawing with his friend x through the whole scale of numbers, to find himself alone, and nailed to a certain value!

Uncle Pen.:—You will not fail to observe that this succeeded only with the supposition that x and y had the same pair of values in both equations: this being supposed, we *eliminated*, i. e. *expelled*, x. You will have occasion, if you pursue this study, to admire still more the devices of *elimination.*

It *may* happen that $B'A = BA'$ in (C): this gives $Y =$ infinite, and X also: so that the point of intersection is at an infinite distance. Show from this equation and from (A) and (B), by division, that these lines are parallel to the same line, when $B'A = BA'$.

LESSON IV.

20. Let $ABba$ be any rectangle, whose base is Aa, and altitude ba: and let $ACca$, $AC'c'a$, be any parallelograms having the same base Aa, and lying within the same parallels, Aaf, Bbc.



Def. The altitude of a parallelogram or of a triangle is the perpendicular let fall from an angle on the base or base produced.

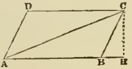
As $Cf = cf' = AB$, by Prop. C. [2], AB is the common altitude of the three parallelograms.

Because $Bb = Aa = Cc = C'c'$, by [2] Prop. C, $BC = bc$, and $Bc' = bC'$; and since $ab = AB$, and the angle $ABC = abc$, the triangle abc will exactly cover the triangle ABC; and ABc' will cover abC'. Denoting triangles by three and quadrilaterals by four letters;

$$ABC = abc,$$
$$ABC - bdC = abc - bdC,$$
$$ABdb = Cdac,$$
$$ABdb + Ada = Cdac + Ada,$$
$$ABba = CAac.$$

$$ABc' = abC',$$
$$C'BAa = C'BAa,$$
then by addition,
$$ABba = C'Aac'.$$

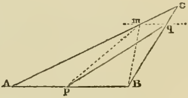
A. *Parallelograms* (CAac and C′Aac′), *which have the same base and altitude, are equal, being each equal to* (ABba) *the product or rectangle of that base and altitude:* vid. definition in (14).



Let ABC be any triangle, whose base is AB. If AD parallel to BC be drawn to meet CD parallel to AB, the sides AD and DC are in order equal to CB and BA, and contain the angle $ADC =$ the angle CBA by [2] Prop. C; and the triangles ABC and ADC will exactly cover each other, and are equal.

Hence the triangle ABC is half the parallelogram $ADCB$, and its area is half the product of AB and CH, CH, $\perp$ on AB, being the altitude either of ABC, or of $ADCB$. We have thus proof that,

B. *The area of a parallelogram is the rectangle (or product) of its base and altitude; and the area of a triangle is half the product of its base and altitude.*



Prob. To divide a triangular plot of ground ABC, whose area is H square feet, into halves, by a line through p a point of the side AB.

Let $pB = h$ feet, pB being not less than pA; then if we suppose the thing done, and that pq is the line required to be drawn, we know that $h \times \frac{1}{2}$ (altitude of q from base AB) is the area of pqB; call this altitude y; we have the condition $\frac{1}{2} \,.\, hy = \frac{1}{2} \,.\, H$; giving $y = H{:}h$, by division of equals by $\frac{1}{2}h$. This fraction $H{:}h$ is the number of feet, y, in the $\perp$ distance of q from the base. If we can draw a $\perp$ through A, p, or B, of the length y feet, and through its extremity a parallel to AB, cutting AC in m and BC in q, either of the triangles pqB or pmB will contain half the area of ABC.

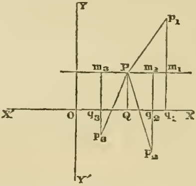
Richard:—I think the shortest way to do this would be to make a right angle at A and another at B, by the device

that Pythagoras, as I suppose, first taught to the bricklayers; then measuring y feet on my two perpendiculars, I should readily draw the line mq.

Uncle Pen.:—You will observe that an *area* is always given in *square units*; and if I speak of the area H, I always mean H *square inches*, unless a different measure is expressly named. The value of y would be perhaps more luminously written $12y \,.\, I = H \,.\, (12)^2 I^2 : (h \,.\, 12I)$, where I is our linear unit, as in (14).

21. Let the point

P be $(x = l,\ y = a)$, and p_1 be $(x_1 = Oq_1,\ y_1 = m_1 q_1)$,
p_2 be $(x_2 = Oq_2,\ y_2 = -q_2 p_2)$, and p_3 be $(x_3 = Oq_3,\ y_3 = -q_3 q_3)$,



the co-ordinates being *rectangular*, parallel to the *right axes* OX and OY.

$$Pm_1 = Qq_1 = Oq_1 - OQ = \qquad x_1 - l;$$
$$p_1 m_1 = p_1 q_1 - m_1 q_1 = p_1 q_1 - PQ = y_1 - a;$$

whence by [7], the triangle $Pp_1 m_1$ gives

$$(x_1 - l)^2 + (y_1 - a)^2 = (Pp_1)^2;$$

Read, the squared difference (x at 1 minus l) + the squared diff. $(y_1 - a)$ equals $(Pp_1)^2$. The subindices under x and y prevent confusion among p_1, p_2, p_3.

$$Pm_2 = Qq_2 = Oq_2 - OQ = x_2 - l;$$
$$p_2 m_2 = m_2 q_2 + q_2 p_2 = PQ + q_2 p_2 = a - y_2;$$
$$\text{for } y_2 = -q_2 p_2, \text{ and } + q_2 p_2 = -y_2.$$

Also,

$$(x_2 - l)^2 + (a - y_2)^2 = (Pp_2)^2, \text{ by [7], taking the } \triangle\ Pp_2 m_2,$$
$$Pm_3 = Qq_3 = OQ - Oq_3 = l - x_3,$$
$$p_3 m_3 = m_3 q_3 + q_3 p_3 = PQ + q_3 p_3 = a - y_3; \text{ for } y_3 = -q_3 p_3.$$
$$(l - x_3)^2 + (a - y_3)^2 = (Pp_3)^2, \text{ by [7], } \triangle\ Pp_3 m_3.$$

Now in the value of $(Pp_2)^2$, $(a-y_2)^2=(y_2-a)^2$;

$$[(a-y_2).(a-y_2)=a^2-ay_2-y_2a+y_2^2=(y_2-a).(y_2-a)]$$

which is merely asserting as in [3] (5) that $-n\times -n=n\times n$.

Hence
$$(Pp_1)^2=(x_1-l)^2+(y_1\div a)^2=(Pm_1)^2+(m_1p_1)^2,$$
$$(Pp_2)^2=(x_2-l)^2+(y_2-a)^2=(Pm_2)^2+(m_2p_2)^2,$$
$$(Pp_3)^2=(x_3-l)^2+(y_3-a)^2=(Pm_3)^2+(m_3p_3)^2;$$

and generally, if r be the distance required between the point (l, a) and *any given point* (x, y),

$$\text{(D)} \qquad r^2=(x-l)^2+(y-a)^2,$$

is the square of the distance, if for x and y be put their proper values *with their proper signs*.

If (x, y) is the point $(3, -4)$, $r^2=(3-l)^2+(-4-a)^2$. If (x, y) is the point $(-3, 4)$, $r^2=(-3-l)^2+(4-a)^2$; if (x, y) is $(-3, -4)$, $r^2=(-3-l)^2+(-4-a)^2$, or $=(3+l)^2+(4+a)^2$ which is the same thing, because $m^2=(-m)^2$ by [3]. If then r in (D) be constant, like l and a, and (x, y) be a variable point, the equation affirms that the distance between (l, a) and (x, y) is always r; this is true of all the points in the circle whose centre is (l, a) and whose radius is r, and of no other points: *therefore* (D) *is the equation to a circle, whose centre is* $(x=l, y=a)$ *and whose radius is* r *inches.*

This equation may be written in any of the forms, *v.* (22),

$$\left.\begin{aligned} (x-l)^2+(y-a)^2&=r^2,\\ x^2-2xl+l^2+y^2-2ya+a^2&=r^2,\\ x^2+y^2-2lx-2ay&=r^2-a^2-l^2. \end{aligned}\right\} \qquad \text{D}$$

You should satisfy yourself by varying the position of P in the angles about O, that (D) gives exactly the square distance between P and (xy). If, for instance, P were the point $(-\cdot 3, \cdot 4)$,

$$(x+\cdot 3)^2+(y-\cdot 4)^2=r^2$$

is the equation of the circle whose centre is $(-\cdot 3, \cdot 4)$, and radius r, l in (D) having in this the value $-\cdot 3$, and P being in the angle $X'OY$.

Required the distance r *between the points* (2, 3) and $(\cdot 3, -\cdot 4)$, *referred to right axes.* By (D) we have

$$r^2=(2-\cdot 3)^2+(3+\cdot 4)^2=(1\cdot 7)^2+(3\cdot 4)^2,$$

whence, extracting the square roots of these equal quantities,

$$r = \pm \sqrt{(1\cdot 7)^2 + (3\cdot 4)^2},$$

$$r = \pm \sqrt{2\cdot 89 + 11\cdot 56} = \pm \sqrt{14\cdot 45} = \pm 3\cdot 80131556.$$

The sign $\pm$ is introduced generally in the solution of such an equation as $r^2 = N$, thus, $r = \pm \sqrt{N}$, to show that r may have either sign you choose to take; for since $m \times m = m^2$, and $-m \times -m = m^2$, m^2 has either m or $-m$ for its square root, and 14·45 has either $+ 3\cdot 80131556$ or $- 3\cdot 80131556$ for its square root; and 1 has either $+1$ or -1 for its square root: either root squared will give the same number.

Every positive number has two square roots, which differ only in sign.

Richard:—But how can $r = \pm 3\cdot 80131556$ be a true answer to your question? Can there be two distances between (2, 3) and (·3, − ·4), one of them less than nothing?

Jane:—The sign $\pm$ has just been explained to signify that r *may* have either sign: you can choose which you please.

Uncle Pen.:—There are many questions which can be solved only by the extraction of the square root, in which the *negative* root is *inapplicable to the problem:* yet here, if in measuring the distance between two points *you take the sign into account at all,* the length between the point P and the point P' has as much right to one sign as to the other; for if the direction is positive from P to P', it must of course be negative from P' to P. From D follows

$$r = \pm \sqrt{(x - l)^2 + (y - a)^2}. \qquad \text{(D}')$$

If a and b are the sides of any right-angled triangle, and c the hy*poth*enuse, $c^2 = a^2 + b^2$, and $c = \pm \sqrt{a^2 + b^2}$: call this *radical*, or *surd*, *poth. ab*: and dismissing all thoughts of triangles and lines, let poth. ab. be our abbreviation of *the arithmetical square root of the sum of the two squares a^2 and b^2; poth. ab.* is *a number*, viz. of the inches in the hypothenuse of a right-angled triangle, whose sides are the lengths a and b in inches. To find the distance R between two points (x_0, y_0) and (x_1, y_1) whose *rectangular* co-ordinates are given, we have the following rule, $R = \pm \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$; or *to* $(\mathrm{x}_1 - \mathrm{x}_0)^2$, *the squared algebraic difference of the* x's, *add* $(\mathrm{y}_1 - \mathrm{y}_0)^2$, *the squared algebraic difference of the* y's; *the square root of this sum is the distance required.* Equation (D′) says,

[12] Pŏth.{Dí(xl) Dĭ($y\acute{a}$)}, joins($x'y$) to (la), in co-ords. rĕctā. — pron. dixle. (ya) a monosyl. xy a dissyl.

i.e. the square root of the sum of the two squares, $\{Diff. (x-l)\}^2$ and $\{Diff. (y-a)\}^2$ joins the point (x, y) to the point (l, a), given in *rect*-angular *co-or*dinates.

Observe that the *algebraic difference* $(x-l)$ may be an arithmetical *sum*, if x and l have different signs: thus if $x = 2$, $l = -3$, $(x-l) = 2+3$; if $x = -1$, $l = 5$, $(x-l) = -(1+5)$, a negative sum.

Let DUQ, (pron. duck) stand for *du*o *q*uadrata, two squares, then

(D) $(x-l)^2 + (y-a)^2 = r^2 = rr$, is a given circle.

[13] DUQ\{Dí(xl) Dĭ.($y\acute{a}$)\} Is $r\ r'$, (in Recta) Gives círc. cĕn ($l\acute{a}$): — pron. dixle; Di = difference; rr a dissyll.; *is* for =; *la* a dissyll.

i.e. *Du*o *q*uadrata $\{Diff. (x-l)\}^2$ and $\{Diff. (y'-a)\}^2$ is rr; gives, *in rect*angular co-ordinates, a circle whose *cen*tre is (l, a); it is needless to add that r is the radius.

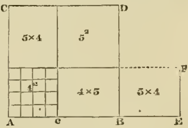
22. The following are of great importance to be remembered.

$$(a+b)^2 = (a+b).(a+b) = a^2 + ab + ba + b^2 = a^2 + b^2 + 2ab \quad \text{(a)}$$

$$(a-b)^2 = (a-b).(a-b) = a^2 - ab - ba + b^2 = a^2 + b^2 - 2ab \quad \text{(b)}$$

$$(b+a)(b-a) = \qquad b^2 - ba + ab - a^2 = b^2 - a^2. \quad \text{(c)}$$

From (a) we learn that the square of the sum of any two numbers is the two squares of the numbers + twice their product. Thus $(4+5)^2 = 16 + 25 + 40 = 81$. And we may write the same truth thus, I being the linear unit, $(4I+5I)^2 = 16I^2 + 25I^2 + 2 \times 20I^2$; which is evident to the eye, if $ABDC$ be 9^2, $Ae = 4$. From (b) we learn, that the square of the difference of two quantities is the sum of the two squares, *minus* twice their product.



Thus, $(9-4)^2 = 9^2 + 4^2 - 2 \times (9 \times 4)$, as is visible in the figure; for 5^2 = the whole 9^2, + 4^2 *additional* in the corner, diminished by $(4^2 + 4 \times 5)$ twice taken. In (c) we are informed, that the sum of two numbers, multiplied by their difference, comes to the difference between their squares. Thus $(9+5)(9-5) = 9^2 - 5^2$. If $BE = 5$, the rectangle $AE \times EF = (9+5) \times (9-5) = 9^2 - 5^2$, evidently.

Let Qua. be *qua*drate or squared; let *SorD*, a monosyll., mean *S*um *or D*ifference; SorD(ab), Sum or Diff. of the two numbers (a and b); let $\pm$ be *mol. mo*re or *l*ess; let *le* be *le*ss or minus; for DUQ vid. [13.] We can express (a) and (b) together, $(a \pm b)^2 = a^2 + b^2 \pm 2ab$.

[14] (a) (b) QuăSŏrD($\acute{a}b$) is DUQ($\acute{a}b$) mol two($\acute{a}b$):

ab a monosyl.

(c) Súm($b\breve{a}$) . Dĭ.($b\acute{a}$) is Sq′.b le sq. $\acute{a}$.

pron. squibble squa.

The first line is (a) and (b). Square (Sum or Diff. of a and b) is the two squares of a and b, $\pm 2ab$; ab meaning *always* $a \times b$, unless otherwise indicated; + goes with Sum, − with Diff. Sum (ba) means sum of b and a, or $(b + a)$. *Sq.b* is *sq*uared b; *sq.a* is *sq*uared a. The second line is (c). Sum(ba).Di(ba), written and uttered *together*, is Sum (ba) *times* Diff. (ba).

23. When $l = a = 0$ and $r = 1$, we have from (D, 21)

$$x^2 + y^2 = 1^2,$$

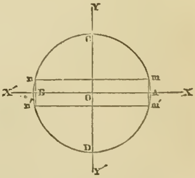
the circle whose radius is unity and centre the origin. We deduce

$$x^2 = 1 - y^2;\ y^2 = 1 - x^2;$$

then, extracting square roots of equals,

$$x = \pm\sqrt{1 - y^2};\ y = \pm\sqrt{1 - x^2}.$$

When $y = 1$, $y^2 = 1$ and $x = 0$, and the same when $y = -1$, which happens at C and D: when $x = +1$ or -1, $x^2 = 1$ and $y = 0$; which is the case at A and B. For every value of $y^2 < 1$, x is *either* square root of a positive number.



Thus $y = \frac{1}{3}$ gives $x = \pm\sqrt{1 - \frac{1}{9}}$ $= \pm\sqrt{\frac{8}{9}} = \frac{\pm\sqrt{8}}{3} = \pm \cdot 942809$; as at m and n, supposing $Am = Bn = \frac{1}{3}$. The same values of x arise from putting $y = -\frac{1}{3}$. When $y > 1$, or < -1, $y^2 > 1$, and $x = \pm\sqrt{1 - y^2}$, is then *the square root of a negative number, an imaginary quantity.* Let $y = 1 \cdot 1$; x^2 then $= 1 - 1 \cdot 21 = - \cdot 79$. What number squared will come to $-\frac{79}{100}$? No possible number; for $m \times m$, whatever sign m may have, must be $+ m^2$.

This imaginary value of x, when $y^2 > 1$, shows that there is no point of the locus at which $y > 1$, or $y < -1$.

Equation (D) gives by extracting roots of equals, after transposition,

$$(x-l)^2 = r^2 - (y-a)^2,$$

$$x - l = \pm\sqrt{r^2 - (y-a)^2}, \text{ or transposing } -l,$$

$$x = l \pm \sqrt{r^2 - (y-a)^2}; \text{ and similarly}$$

$$y = a \pm \sqrt{r^2 - (x-l)^2}.$$

Let the circle, $(x + \cdot 5)^2 + (y-3)^2 = 0\cdot 49$, be given for consideration. We see at once that $(-\cdot 5, 3)$ is the centre, and that $0\cdot 7$ is the radius. We deduce

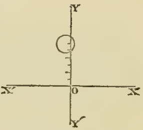
$$x + \cdot 5 = \pm\sqrt{\cdot 49 - (y-3)^2}, \text{ or}$$

$$x = -\cdot 5 \pm \sqrt{\cdot 49 - (y-3)^2}; \text{ and}$$

$$(y-3) = \pm\sqrt{\cdot 49 - (x+\cdot 5)^2}, \text{ or}$$

$$y = 3 \pm \sqrt{\cdot 49 - (x+\cdot 5)^2}.$$

If $y = 0$, $x = -\frac{1}{2} \pm \sqrt{\cdot 49 - (-3)^2} = -\frac{1}{2} \pm \sqrt{-8\cdot 51}$, a negative real number, added to an *imaginary quantity*; such an x has no existence, and this shows us that there is *no point* of the circle at which $y = 0$, or that it *nowhere* meets the axis of x.



$y = 3$ gives $x = -\cdot 5 \pm \cdot 7$,

either $\cdot 2$ or $-1\cdot 2$,

$x = 0$ gives $y = 3 \pm \sqrt{\cdot 24}$, either $3\cdot 4899$, or $2\cdot 5101$,

$x = -\frac{1}{2}$ gives $y = 3 \pm \cdot 7$ either $3\cdot 7$ or $2\cdot 3$.

If we *untie* in (D), we obtain by [14] (Qua D (ab)),

$$x^2 - 2lx + l^2 + y^2 - 2ay + a^2 = r^2.$$

If we put for x any given value x_1, this is, by transposing,

$$y^2 - 2ay = r^2 - x_1^2 + 2lx_1 - l^2 - a^2, \text{ or,}$$

putting N for the *known number* in the right member, $y^2 - 2ay = N$, from which we have to find the y corresponding to $x = x_1$.

It is very convenient to make such abbreviations, for we can at any moment find the value of N, by adding r^2 to $2lx_1$, and then subtracting $(x_1^2+l^2+a^2)$ from their sum: this is all known; for l, a, r, and x_1, are *given*. Innumerable questions of great beauty and interest reduce themselves to a result of this form,

$$y^2-2ay=N,$$

in which a and N are *known*, and y is the number whose value is *to be found*. This is called *a quadratic equation*, containing y^2, *y quadrate*, the *square* of the unknown quantity. To solve this, that is, to find from it the value or values of y, add a^2 to the equals, thus making sums still equal;

$$y^2-2ay+a^2=N+a^2.$$

The left side is '*DUQ.ay* less $2ay$,' and is therefore by [14], '*Qua D.(ay)*,' or $(y-a)^2$; i.e.

$$(y-a)^2=N+a^2;$$ whence extracting

the square roots of these equals

$$y-a=\pm\sqrt{N+a^2},$$ or, adding a to equals,

$$y=a\pm\sqrt{N+a^2};$$

whereby y is given in known quantities, and has plainly two values, which can be found by extracting the square root of the number $(N+a^2)$. This is the result before obtained; for putting for N its value,

$$y=a\pm\sqrt{r^2-(x_1^2-2lx_1+l^2)-a^2+a^2}$$ or by [14],

$$y=a\pm\sqrt{r^2-(x_1-l)^2}.$$

24. If then y be any quantity of which we are in search, and we know that the square of y, less twice the product of y and any known number a, is equal to any known number N, we obtain two *values* of y, called the *roots of the equation* $y^2-2ya-N=0$, by simply adding in turn to a the square roots, positive and negative, of (a^2+N). This may be thus remembered, putting sq.y and asq. (pron. squī and ask) for *sq*uared *y* and *a* *sq*uare.

[15] If *sq.y'* le twŏ *yá* be N, *ya* a monosyl.

y's *á* mol RóoM *ăsq N'*, vid. mol. le, in [14].

M, like S, will often stand conveniently for SuM. RooM is the square *Root* of the su*M* of the two indicated quantities (a^2, N). If N is negative, RooM is in fact the root of a difference.

If you should forget the proof of [15], reason thus for a moment upon it. If the assertion $y = a \pm \sqrt{a^2 + N}$ be true, then must $y - a = \pm \sqrt{a^2 + N}$, by subtraction from equals: and the squares of the equals last written must be equal, or $(y-a)^2 = a^2 + N$, which is by [14] 'QuaD ab' (or QuaD ya), $y^2 - 2ay + a^2 = a^2 + N$, whence $y^2 - 2ya = N$, as it ought to be.

Thus the equation $3t^2 - 5t - 11 = 0$, is by transposition and division

$$t^2 - \frac{5}{3}t = \frac{11}{3}; \text{ of the } \textit{form } t^2 - 2at = N,$$

t is here the unknown quantity y; $a = \frac{5}{6}$, $N = \frac{11}{3}$, whence

$$t = \frac{5}{6} \pm \sqrt{\frac{25}{36} + \frac{11}{3}}.$$

Now $\frac{25}{36} + \frac{11}{3} = \frac{25 + 12 \times 11}{36} = \frac{157}{36} = \frac{157}{6^2}$, so that

$$t = \frac{5}{6} \pm \sqrt{\frac{157}{6^2}} = \frac{5 \pm \sqrt{157}}{6};$$

for you know by arithmetic, that the square root of a fraction is the quote of the roots of its numerator and denominator. Before the solution of a quadratic can be obtained by the formula [15], it is necessary that the square of the unknown should have the coefficient unity; and we accordingly had to divide the above equation by 3. From

$$at^2 - et = c \text{ comes}$$

$$t^2 - \frac{e}{a}t = \frac{c}{a}, \text{ whence by [15]}, \qquad \left(\frac{e}{2a} \text{ for our } a\right)$$

$$t = \frac{e}{2a} \pm \sqrt{\frac{e^2}{4a^2} + \frac{c}{a}} = \frac{e}{2a} \pm \frac{\sqrt{e^2 + 4ac}}{2a},$$

$$\text{for } \frac{e^2}{4a^2} + \frac{c}{a} = \frac{e^2}{4a^2} + \frac{4a \cdot c}{4a \cdot a} = \frac{e^2 + 4ac}{(2a)^2}$$

LESSON V.

25. *Required a number such that the sum of it and its reciprocal shall be 3.* We can *talk about* the number even while it is unknown, if we give it a name: call it the number y. Then it must be true that

$$y + \frac{1}{y} = 3, \text{ or multiplying equals by } y,$$

$$y^2 + 1 = 3y, \text{ or transposing 1 and } 3y,$$

$$y^2 - 3y = -1 \text{ of the form } y^2 - 2ay = N;$$

here our $a = \frac{3}{2}$ and $N = -1$, so that by [15],

$$y = \frac{3}{2} \pm \sqrt{\frac{9}{4} - 1} = \frac{3}{2} \pm \sqrt{\frac{9-4}{4}} = \frac{3}{2} \pm \frac{\sqrt{5}}{2}$$

$$= \frac{3+\sqrt{5}}{2} \text{ or } \frac{3-\sqrt{5}}{2}, \text{ i. e. } = 2{\cdot}618034 \text{ or } 0{\cdot}381966.$$

You may take either of these values for the number sought; the other is its reciprocal; and you see that $2{\cdot}618034 + {\cdot}381966 = 3$. At first sight you would not have suspected that these were reciprocal numbers; but we can easily test the matter.

If $\frac{3+\sqrt{5}}{2} = \frac{2}{3-\sqrt{5}}$, it must follow that

$3 + \sqrt{5} = \frac{4}{3-\sqrt{5}}$, and multiplying equals by $3 - \sqrt{5}$, [14 c],

$$3^2 - (\sqrt{5})^2 = 4 \text{ or } 9 - 5 = 4; \text{ which is true.}$$

I prefer the more general question, To find a number such that m times the number added to n times its reciprocal shall give a sum equal to r. And you shall choose your own values for m, n, and r. All that I have to do is to say

$$my + n\frac{1}{y} = r, \quad \text{then as before multiplying by } y,$$

$$my^2 + n = ry, \quad \text{whence by transposition,}$$

$$my^2 - ry = -n, \quad \text{and dividing equals by } m,$$

$$y^2 - \frac{r}{m}y = -\frac{n}{m}.$$

Here $a = r{:}2m$, $N = -n{:}m$, and by [15]

$$y = \frac{r}{2m} \pm \sqrt{\frac{r^2}{4m^2} - \frac{n}{m}} = \frac{r \pm \sqrt{r^2 - 4mn}}{2m}.$$

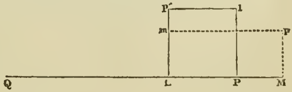
If you fix on the values $r = 3$, $m = 1 = n$, you have the preceding problem and solution. If you choose $m = 1 = n$, $r = 2$, you obtain $y = \frac{2 \pm \sqrt{4 - 4}}{2} = 1 \pm 0$. Here is but one value of y, which however solves the problem; for unity added to its reciprocal gives 2. If you choose $m = 2$, $n = 2$, $r = 4$, you obtain $y = 1 \pm 0$ again; and it is true that $2 \cdot 1 + 2{:}1 = 4$. If $r = 8$, $m = 8$, and $n = 2$, you obtain $y = \frac{1}{2}$; and it is true that $m \,.\, \frac{1}{2} + n \,.\, 2 = r$.

In all these instances the radical vanishes, because $r^2 = 4mn$, and there is but one value for y, which is $r{:}2m$. If you take the case of $m = r = n = 1$, you obtain

$$y = \frac{1 \pm \sqrt{1 - 4}}{2} = \frac{1 \pm \sqrt{-3}}{2},$$

two imaginary values, for $\sqrt{-3}$ has no real existence. This shows that there is no number such that $y + \frac{1}{y} = 1$, or such that the sum of it, and its reciprocal, is unity. If $r = 1$, $m = 1$, $n = -1$, $y = \frac{1 \pm \sqrt{1^2 + 4}}{2} = \frac{1 + \sqrt{5}}{2}$ or $\frac{1 - \sqrt{5}}{2}$, i. e. $= 1{\cdot}618034$ or $-0{\cdot}618034$. Either of these added to -1 times its reciprocal, gives unity for the sum.

Given LM *a line* A *inches in length; it is required to divide it in a point* P, *so that the rectangle under the line and one part shall be equal to the square of the other part.*



Let the greater segment LP of the line be y inches in length; the other will be $A - y$ inches. And if I be the linear unit, the condition is $(LP)^2 = (LM) \times (PM)$, or

$$(yI)^2 = AI \cdot (A - y) \cdot I,$$

i. e. $y^2I^2 = A^2I^2 - AyI^2$, whence by division of equals,

$$y^2 = A^2 - Ay, \text{ or}$$

$$y^2 + Ay = A^2, \text{ which is}$$

$y^2 - 2ya = N$, if $a = -\frac{1}{2}A$ and $N = A^2$; we have then [15]

$$y = -\frac{A}{2} \pm \sqrt{\frac{A^2}{4} + A^2} = -\tfrac{1}{2}A \pm \frac{\sqrt{5A^2}}{\sqrt{4}} = \frac{-A \pm \sqrt{A^2}\sqrt{5}}{2}$$

$$= \frac{-A \pm A\sqrt{5}}{2} = A \cdot \tfrac{1}{2}(-1 \pm \sqrt{5}).$$

If $A = 1$, $y = \dfrac{1}{2}(-1 \pm 2{\cdot}236068) = {\cdot}618034$ or $-1{\cdot}618034$.

These are the roots of $y^2 + y = 1$; those of $y^2 - y = 1$, found above, differ from them only in sign. If $LM = 1$, and $LP = {\cdot}618034$, the square on LP ($LPlp'$) is equal to the rectangle $LM \times MP$ ($LMpm$); (supposing $LP' = LP$, and $Mp = MP$.) There is a second value of y, and therefore a second point, (Q) in which the line LM can be cut so as to solve the problem; but the distance of Q from L is of the opposite sign to that of LP, and must be measured in the opposite direction. If $LQ = -1{\cdot}618034$; the square of LQ = the rectangle $LM \times MQ$. The segments made are in either case measured from the point of section to the extremities of the given line PL, PM, and QL, QM. This is Prop. 11 of the second book of Euclid's elements. You will find no difficulty in proving any of the 10 preceding propositions. Thus the 10th merely affirms that

$$(2a + b)^2 + b^2 = 2(a + b)^2 + 2a^2,$$

the truth of which is evident if you attend to [14] about QuaS(ab).

Jane:—Richard solved mentally last week this question: To find a number such that the square of it added to its half shall = 18. Suppose 17 put in the place of 18; could you solve the question?

Uncle Pen.:—It is too easy only. I prefer finding a number y, such that,

$$my^2 + n \cdot \frac{y}{2} = c,$$

i.e. such that m times its square + n times its half shall be

equal to c; m, n, and c, shall have any values you may think of. Dividing these equals by m,

$$y^2 + \frac{n}{m} \cdot \frac{y}{2} = \frac{c}{m}; \text{ which is}$$

$$y^2 - 2ya = N,$$

if $a = -\frac{1}{2} \cdot \frac{n}{2m} = -\frac{n}{4m}$, and $N = \frac{c}{m}$. Hence by [15],

$$y = -\frac{n}{4m} \pm \sqrt{\frac{n^2}{16m^2} + \frac{c}{m}}.$$

In your question, $m = 1$, $n = 1$, $c = 17$; so that by substitution of these values,

$$y = -\frac{1}{4} \pm \sqrt{\frac{1}{16} + 17} = -\frac{1}{4} \pm \sqrt{\frac{1+272}{16}} = \frac{-1 \pm \sqrt{273}}{4}$$

$$= +3\cdot8806779, \text{ or } -4\cdot3806779,$$

either of which numbers, squared and increased by half itself, will give 17. Richard's answer was 4; let him find the other value of y, corresponding to $c = 18$, and he will have *completed* his solution.

If you demand a number such, that *thrice* its square, *diminished* by *four-fifths* of its half shall be *two;* we have

$$m = 3, \quad n = -\frac{4}{5}, \quad c = 2;$$

and we obtain from the same expression,

$$y = \frac{\frac{4}{5}}{12} \pm \sqrt{\left(\frac{1}{15}\right)^2 + \frac{2}{3}} = \frac{1}{15} \pm \sqrt{\frac{151}{15^2}} = \frac{1 \pm \sqrt{151}}{15}$$

$$= 0\cdot8858803, \text{ or } -\cdot752547.$$

To verify this,

$$3 \times (\cdot8859)^2 = 2\cdot35445643, \text{ and } \frac{4}{5} \text{ of } \frac{\cdot8859}{2} = \cdot35436;$$

the difference of these $= 2\cdot0009643$, somewhat too high, because we called the first value of y $\cdot8859$, nearly two hundred thousandths too great. Also,

$$3 \times (-\cdot7525)^2 = 1\cdot69876875, \text{ and } \frac{4}{5} \text{ of } \frac{-\cdot7525}{2} = -\cdot301,$$

the difference of these is $1\cdot69876875 + \cdot301 = 1\cdot99976875$, somewhat too low, because we took our second value of y too great, i.e. too near zero, by nearly half a ten thousandth.

26. Thus *every quadratic has two roots:* if the quantity under the radical $\sqrt{a^2+N}$ is negative, i.e. if N is negative, and $> a^2$, both roots are imaginary; if the radical is real, both roots are real; and if the radical vanishes, or $a^2+N=0$, which requires $N=-a^2$, both the roots are reduced to the same value $y=a$, which is the only case in which the roots can be equal to each other. If you add together the two roots

$$y_1 = a+\sqrt{a^2+N}, \text{ and } y_2 = a-\sqrt{a^2+N},$$

you obtain $y_1+y_2=2a$, which *with a changed sign* is the *coefficient* or multiplier of y in the equation

$$y^2-2ay=N;$$

if you multiply together the two roots

$$y_1 = a+\sqrt{a^2+N}, \text{ and } y_2 = a-\sqrt{a^2+N},$$

you obtain $y_1y_2=a^2-(a^2+N)$ [14], (sq b le sq a, if b in [14] be put for our a here, and a in [14] for our radical $\sqrt{a^2+N}$), i.e. $y_1y_2=-N$, which is the *absolute term,* (term free from the unknown y) with its proper sign, in the equation

$$y^2-2ay-N=0.$$

In any quadratic equation $(y^2-py+q=0)$, *of which all the terms stand in one member, the coefficient of* (y^2) *the square of the unknown quantity being unity, the coefficient* $(-p)$ *of the first power of the unknown* (y) *is, with a changed sign, the sum of the roots of the equation, and the absolute term* (q) *is the product of those roots.*

[16] If nil be (sq.y' le py' and q') vid. sq.y in [15] and le.
p's sùm, and q is prod. of roo.

nil for zero; *prod*uct *of roo*ts.

27. The equation whose roots are y_1 and y_2, is

$$(y-y_1)(y-y_2)=0,$$

and this is the same with

$$y^2-(y_1+y_2)y+y_1y_2=0,$$

neither of which is satisfied by any values of y except $y=y_1$ or $y=y_2$, while both equations are *visibly* true for either of these values of y.

Hence, before we know the roots of $y^2+6y+7=0$, we are certain that their sum is -6, and their product $+7$.

And we are certain of this before we know whether the equation has any real root at all. Thus the roots of $y^2 - 6y + 10 = 0$ have a sum $= +6$ and a product $= 10$; but from $y^2 - 2 \times 3y = -10$ comes [15] $y = 3 \pm \sqrt{9-10} = 3 \pm \sqrt{-1}$, both *imaginary* values: yet $3 + \sqrt{-1}$ added to $3 - \sqrt{-1}$ gives $+6$; and

$$(3 + \sqrt{-1})(3 - \sqrt{-1}) = by\ [14]$$
$$3^2 - (\sqrt{-1})^2 = 9 - (-1) = 10.$$

Jane:—It is a mystery to me that quantities which are impossible and can have no existence, should yet have real and intelligible properties, should have a real sum, and a real product, exactly like two genuine numbers. It looks so like a contradiction.

Uncle Pen.:—It would be a contradiction, if these two imaginaries were said to have a sum and a product *exactly like* real numbers. Their sum may be real and their product may be real, but they must be *incongruous,* such as cannot possibly exist together as sum and product of *the same two quantities.* If you ask for two numbers whose sum is 6, and whose product is 10, you ask for an impossibility.

You put your question thus: $y^2 - 6y + 10 = 0$; what then is y? And the wonderful oracle of Algebra answers, $y = 3 \pm \sqrt{-1}$: an answer perfectly correct; for if you square either of these values, then subtract 6 times that value, then add 10, the result is zero.

Jane:—So then whatever mystery or appearance of contradiction there may be here, it springs not from the answer of the oracle, but from the ignorance of the interrogator. His duty is not to cavil at the response, but to go away ashamed of himself and wondering.

Richard:—I should like to try the verification just hinted at of the truth of your Pythia's arithmetic. If $y = 3 + \sqrt{-1}$,

$$y^2 = (3 + \sqrt{-1})^2 = 9 + 2 . 3\sqrt{-1} + (\sqrt{-1})^2 = 9 + 6\sqrt{-1} - 1,$$

by [14] 'QuaS,'

$$-6y = -6 . (3 + \sqrt{-1}) = \qquad -18 - 6\sqrt{-1},$$

so that

$$y^2 - 6y + 10 = 9 + 6\sqrt{-1} - 1 - 18 - 6\sqrt{-1} + 10 = 0.$$

She is right.

LESSON VI.

28. *Uncle Pen.:*—Not every equation containing x and y in the second degree represents a circle. Terms of the second degree are Ax^2, Bxy, Cy^2. Terms of the third degree in x and y are Ax^3, Bx^2y, Cxy^2, Dy^3. From equation D (21) we see that in the equation to a circle referred to right axes, x^2 and y^2 have the same coefficients, and no term exhibits the product of the two variables. If then, the axes being rectangular,

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, \text{ is a circle,}$$

$$A \text{ and } C \text{ must be equal numbers, and } B = 0.$$

The circle

(E) $$3x^2 + 3y^2 + 5x + 6y - 7 = 0,$$

is by division and transposition

$$x^2 + y^2 + \frac{5}{3}x + 2y = \frac{7}{3}, \text{ or}$$

adding to these equals the squares of the half-coefficients of x and y,

$$\left(x^2 + 2 \times \frac{5}{6}x + \frac{5^2}{6^2}\right) + (y^2 + 2y + 1^2) = \frac{7}{3} + \frac{5^2}{6^2} + 1^2 = \frac{145}{36},$$

$$\text{or [14], } \left(x + \frac{5}{6}\right)^2 + (y+1)^2 = \frac{145}{36} = \left(\frac{\sqrt{145}}{6}\right)^2.$$

The circle (E) has its centre at the point

$$\left(\frac{-5}{6}, -1\right) \text{ and its radius is } \frac{\sqrt{145}}{6} = 2{\cdot}0069324.$$

If we change $+7$ for -7 in (E) it is no longer the same curve, and the result is

$$\left(x + \frac{5}{6}\right)^2 + (y+1)^2 = \left(\frac{\sqrt{-23}}{6}\right)^2;$$

the radius here is imaginary; therefore *this* circle has no existence, and there is no point (x, y) that will satisfy

$$3x^2 + 3y^2 + 5x + 6y + 7 = 0;$$

i. e. there are no two numbers possible, x and y, such that thrice the sum of their squares, added to 5 times one number + 6 times the other, shall be equal to − 7. Show now that

$$3x^2 + 3y^2 + \cdot 5x + \cdot 6y - 7 = 0$$

is a circle whose centre is

$$\left(-\frac{1}{12}, \ -\frac{1}{10}\right), \text{ and radius} = 1\cdot 5306157,$$

and that this locus becomes impossible if the *absolute term* be $+7$, or if it be any positive number $> \dfrac{61}{1200}$:

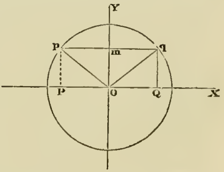
($>$ means *greater than*).

The equation to a circle, referred to rectangular co-ordinates, can always be reduced to the form,

$$(x - b)^2 + (y - a)^2 = r^2,$$

by the method above employed in reducing equation E. Hence *the co-ordinates of the centre, and the radius of the circle, can always be found from its equation.*

29. Look again at the circle whose centre is the origin of *right* axes, and radius $= r$. Its equation (23) is $x^2 + y^2 = r^2$, whence follows $x = \pm\sqrt{r^2 - y^2}$. This affirms that the two values of x determined by any given value of y are equal and of opposite signs. If y be the positive length Om, measured on OY, the corresponding points of the circle are found by drawing pmq parallel to OX. Then the ordinates qQ and pP are each $= Om$, [2]; and the values of x are OQ and OP, the first being $+\sqrt{r^2 - Om^2}$ and the second $-\sqrt{r^2 - Om^2}$. Hence mq and mp by [2] are equal, save the sign: in other words, *the chord* pq *is bisected by the perpendicular* Om *let fall on it from the centre.* This may be *any* chord; for Om is *any* length less than r, and our axis OX may be *any* line through O. Hence we can affirm:



The perpendicular let fall from the centre of a circle upon any chord of it bisects that chord. (a)

$$Ax^2 + Ay^2 + Ey = 0, \text{ or, putting } - T \text{ for } E{:}A,$$

(G) $$x^2 + y^2 - Ty = 0, \text{ is simply}$$

$$(x-0)^2 + (y - \tfrac{1}{2}T)^2 = \tfrac{1}{4}T^2,$$

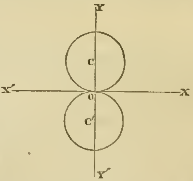
by adding $(\frac{1}{2}T)^2$ to the equal sides.

This is therefore (the axes still rectangular) a circle having its centre at $(0, +\frac{1}{2}T)$, and radius $= \frac{1}{2}T$. By transposition and extraction of equal roots,

$$x = \pm\sqrt{-y^2 + Ty};$$

which shows that T and y must have like signs, otherwise both $-y^2$ and $+Ty$ would be negative, and x impossible. When $+Ty = +y^2$, or $y = +T$, $x = 0$; when $+Ty > y^2$, x has two values, which are equal but of contrary signs: as y approaches to the value zero, the values of x continually approach zero from opposite sides; when $y = 0$, $x = \pm 0$, or the two values of x coincide at zero.

These considerations supplied by the equation, reveal the position of the circle (G): the axis of x, on which lie all the points having $y = 0$ (11), meets the circle (G) only in one point, 0: and the circle will lie entirely above or entirely below OX, according as T is positive or negative. The line OX is said to be a *tangent* at O or *toucher* of the circle, as is every line which meets it in *one point only*. The point O is here *the point of contact*. The ordinate $\frac{1}{2}T, = -\frac{1}{2}E{:}A$, of the centre is perpendicular to OX, and passes through O; and since, by drawing our axes accordingly, OX may be *any* tangent, we can affirm:



The perpendicular on the tangent of any circle let fall from its centre passes through the point of contact; or, *the line drawn from the centre to any point of the circle is perpendicular to the tangent at that point;* or, *the tangent of any circle is perpendicular to the radius of contact.* (b)

To remember this and (G) from which it is proved, say,

[17] DUQ($y'x$) is Ty'
Has tán. nil's y': pron. (wix) vid. DUQ [13]
(a) Rad per'p on chór.'s biChór: pron. tanil's. *is* or *'s* for =.
(b) Toŭch-r'ad is per' on tán:

i.e. the circle $y^2 + x^2 = Ty$, (G) has the tangent $o = y$, or OX, (11); (a) the *rad*ius *perp*endicular *on* any *chor*d is the *bi*sector of the *chord*; (b) the *touch-rad*ius, or rad. of contact, is *per*pendicular on the *tan*gent.

Observe that every curve whose equation has no term free of variables, no *absolute* term, passes through the origin, (0, 0) being evidently in the locus. The equation

H. $$Ay^2 + Ax^2 + By + C = 0,$$

or by division of both sides by A,

$$y^2 + x^2 + (B{:}A)y = -C{:}A,$$

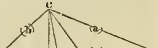
is by addition of $(\frac{1}{2}B{:}A)^2$ to both sides

$$(y + \tfrac{1}{2}B{:}A)^2 + (x+0)^2 = \tfrac{1}{4}B^2{:}A^2 - C{:}A.$$

This is a circle whose centre is in the axis of y at (zero, $-\frac{1}{2}B{:}A$), and whose radius is $\frac{1}{2}(B^2{:}A^2 - 4C{:}A)^{\frac{1}{2}}$. The circle $ax^2 + ay^2 + bx + c = 0$ has its centre in the axis of x.

30. The triangle Opq in the last figure but one, having the equal legs Op, Oq, is called an *isosceles* or *equal-legged* triangle. If it were folded along Om, the perpendicular on the base pq, the portion Oqm would exactly cover Opm; for the angles at m are equal, being right angles, and by [17, a] $mq = pm$. Therefore the base angles Oqp and Opq are equal, as are also the two angles mOp and mOq into which the vertical angle O is divided by the perpendicular on the base. Hence;

In an isosceles triangle the angles at the base opposite to the equal sides are equal; and the perpendicular on the base from the vertex bisects both the vertical angle and the base.



Let a, b, c, be the sides of any triangle; then the angles A, B, C, in order opposite to them, are to be called for ear-memory's sake Ang, Bang, Cang. The perpendicular on the base, Cd, may be called *perc*, *per*. on c: the bisector of the base, Cf, may be styled *bic*, *bi*sector of c; and Ce, the bisector of the vertical angle C, may be called *biCang*, *bi*sector of *Cang*. Then say,

[18] If á's b, An'g is Báng;
and per'c is bic' is bĭCáng.

If in any triangle, $a = b$, then $A = B$; and perc, bic, and biCang are all one and the same line.

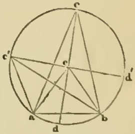
Jane:—It appears to me that the most difficult part of this and all the preceding lessons is *the arithmetic:* the reasoning is all simple enough to follow, if one can only succeed in remembering it all.

Richard:—Do you call the arguments about imaginary quantities so simple? As to the remembering, I have little fear about that: these mnemonics are nothing compared with Greek conjugations and dialects. Try *As in præsenti* and the *verbs in* $\mu\iota$.

Jane:—You have such a taste for hard words, and such a memory for any thing like rhymes, Dickon. By all means astonish the rustics some fine morning by emphasizing these mnemonics along the fields, as you did the other day by spouting Homer!

Uncle Pen.:—If Richard had as much talent for Science as for languages, he would not require these mnemonical aids; no mathematical *genius* needs such assistance. Take some pains to master them, my dear Jane, and you will save your time, of which you have very little to spare for mathematics, a study in which I should be sorry to see you deeply engaged.

Some of the properties which we have been just proving, seem almost evident enough of themselves without learned demonstrations. I shall next introduce you to something which you would have lived long enough without imagining.



31. Let adb be any arc of a circle whose centre is e: and let the chord ab be the common base of any two triangles acb, $ac'b$, having their vertices in the remaining portion of the circumference $ac'cb$. Draw the radii ea, eb, and the diameters dec, $d'ec'$. Then are aeb, $c'eb$, aec', ceb, cea, all *isosceles* triangles. By Prop. D, and [18] the following are true of the four external angles deb, dea, $d'eb$, $d'ea$.

$$deb = ecb + cbe = 2ecb, \qquad d'eb = ec'b + ebc' = 2ec'b,$$

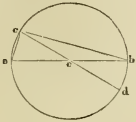
$$dea = eca + eac = 2eca; \quad d'ea = ec'a + eac' = 2ec'a;$$

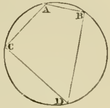
$$\therefore\ deb + dea = 2.(ecb + eca), \quad \therefore\ d'ea - d'eb = 2(ec'a - ec'b),$$

$$\text{or } aeb = 2.acb; \qquad aeb = 2ac'b.$$

That is, since c or c' may be *any* point in the arc $ac'cb$:

(a) *In any circle the angle* (aeb) *at the centre, standing upon any arc* (adb) *is double any angle* (acb) *at the circumference standing on the same arc* (adb).





When the arc *adb* is a semicircle, the angle *aeb* is two right angles, being $= aed + deb$, (Prop. D.) The demonstration about *deb* and *dea* remains still true, and

$$acb = \tfrac{1}{2}\, aeb = \text{a right angle.}$$

(b) *The angle in a semicircle is a right angle.*

(c) *The opposite angles of a quadrilateral inscribed in a circle are supplements.* Vid. Def. 7, (32).

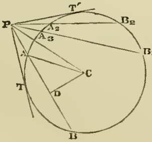
For their sum is that of half the arcs on which they stand; i. e. half the circumference; i. e. two right angles.

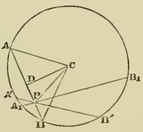
We shall see that the angle *aeb* and the arc *adb* are expressed by the same number: *numerically* the *angle* at the centre upon an arc *is* that *arc*. You may say then,

[19] Rim-angle on arc is arc-démi, (a)
And right is the angle in semi. (b)

i.e. The *angle* at the *rim* (= circumference) standing *on* any *arc*, *is* the *demi-arc*; semi. for semicircle.

This and the following are among the most beautiful of the properties of the circle. Let P be any point without





or within a circle whose centre is C, and let any line PAB through P meet the circle in A and B. Then drawing the radius $CA = r$, and CD perpendicular on AB, we have by [7] in the triangle PCD,

$$(PC)^2=(PD)^2+(CD)^2,$$

$r^2=(AD)^2+(CD)^2\ \{=(CA)^2\}$, whence by subtraction

$(PC)^2-r^2=(PD)^2-(AD)^2$, which by [14, c]

$$=(PD+AD).(PD-AD)=(PD+DB).(PD-AD)$$
$$=(PD+AD).(PD-DB),$$

because $AD=BD$, by [17, a]; that is,

$$(PC)^2-r^2=PA.PB, \qquad \text{(a)}$$

PA being the *sum*, and PB the *difference*, if P is within, and *vice versâ*, if P is without the circle.

As this value of the rectangle or product $PA.PB$ depends only on PC, which is the distance of P, and not in any wise on the length of (CD) that of the line PAB, from the centre, it will remain the same value for every line drawn through P. From P without the circle, $PA.PB=PA_2.PB_2=PA_3.PB_3=(PC)^2-r^2$. From P, within it, $PA.PB=PA'.PB'=PA.PB=+r^2-(PC)^2=-\{(PC)^2-r^2\}$; for PA and PB, both measured from P within the circle, have unlike signs, and their product $(PC)^2-r^2$ given in (a) must have the negative sign. That this number $(PC)^2-r^2$ is negative is very evident, since $r>PC$. Every case may be represented thus,

$$PB.PA=\pm\{(PC)^2-r^2\}; \qquad \text{(a)}$$

taking the upper sign when P is without, and the lower when P is within the circle. If P is neither within nor without, (a) is still true, being then $0=0$. The product $PB.PA$ changes sign in passing through zero.

Let the distance (PC) of any *p*oint from the *c*entre be called *poc*, (*p*oint...C); let *wing* mean *radius* for rhyme's sake, (it flies when revolving); let *Seg.S₁eg in P ring* stand for (*segment* × *segment in* any line through P cutting the *ring*); each segment being measured from P to the ring; and let SUD stand for SUm × Difference, or Difference of squares, of two indicated quantities; thus SUD (ax) is $a^2-x^2=(a+x).(a-x)$, [14, c] SUD (by) is (b^2-y^2): SUD (*poc* wing) $=(poc)^2-r^2$. Then you may say to yourself,

[20] (Seg . S,ég) in P – ríng, pron. sĕgség. P ring a dissyl.
Is mŏl SU'D (*poc* wi'ng), mol = ±.
For P oút or in ring. Pron. pout; out with +, in with –.

Remember that the two segments in (Seg. S_1eg) are both measured from P in all cases. When P is without, the points A and B, or A_2 and B_2, may approach very near each other, till they meet in a point T, or (T'), which is then the point of contact of the tangent PT, and $PA_2 \times PB_2 = (PT')^2$, and $PA \,.\, PB = (PT)^2 = (PT')^2$, whatever be the *secant* or cutter PAB. The same thing is evident from the equation $PA \times PB = (PC)^2 - r^2$, which is $(PT)^2$, since, PTC being a right angle [17, b], $(PC)^2 = (PT)^2 + r^2$. Thus is established that,

The rectangle under the two segments cut off by a circle from a point P *on any line through that point has a constant value, which is the product of the sum and difference of the radius and distance of the point from the centre.*

The rectangle under the two segments of any secant of a circle, both measured from a point P *without it, is equal to the square of the tangent from that point.*

The two tangents to a circle from any point P *are equal.*

LESSON VII.

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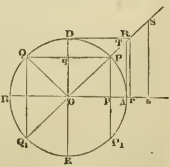
32. As I have undertaken to show you the shortest way to practical and applied Geometry, I shall defer all further treatment of lines and circles, till I have taught you something about the arithmetic of angles. An angle, before it can be made matter of calculation, must be reduced to a number. When the unit of length is once given, *to every number, positive or negative, corresponds a distinct angle, and every angle has its own number.*

Let the circle whose radius (OA) is unity be drawn: then choosing any diameter BOA from which arcs are to be measured, any arc AP determines an angle AOP at the centre standing upon that arc: and it is *sufficiently evident from the perfect symmetry of the circle,* that on every arc of it, as BQ, which is equal to AP, there stands at the centre an angle BOQ equal to AOP. I shall not think it necessary to demonstrate, that *equal arcs of the*



same circle subtend at the centre equal angles. The radius being one inch, *the length of the arc in inches is the number of the angle standing thereon at the centre.* So nearly connected are the arc and the corresponding angle, that we shall often put one for the other, and not hesitate to speak of the angle 2, or the arc 2, indiscriminately; for no confusion can possibly hence arise. If I speak of the angle 1, you are to understand that angle POA at the centre, which is made between OA and the moveable unit radius OP, when P has described from A one inch of *this* circumference in the direction ADB. The angle -1, is that exhibited at the centre when the point P of the moveable radius has swept over one inch from A *in the opposite direction* AEB. The angle 10, (or -10) is that standing at the centre between AO and OP after P has described on *this* circumference from A, in the positive, (or negative) direction round the circle, the length of 10 inches; and so on for the angle 100, or 1000, positive or negative. One angle or arc may thus contain more circumferences of our circle than one; if π (pronounce pi) be the number of inches in our half circumference, 2π is the whole, or four right angles, the whole angular space once round O; π is two right angles, and $\frac{1}{2}\pi$ is one.

If AP be θ inches (pron. thēta) in length, OP will stand at the same point for any of the angles θ, $2\pi+\theta$, $4\pi+\theta$, $6\pi+\theta$, $2n\pi+\theta$; n being any whole number, zero included: θ and $2\pi+\theta$ present the same angular opening to the eye between OA and OP, but are not the same angle, because they cannot be described in the same time by equable motion, nor are they denoted by the same numbers. The number π, like $\sqrt{2}$ and all incommensurable numbers, can never be exactly found: you will learn shortly how to determine π, and at present may take for granted that $\pi = 3{\cdot}1415926536$. I recommend you always to call this *tafalout**, after the fashion of Dr Grey, the prince of mnemonicians. Every number less than *tafalout* gives an angle less than

* More at length π is = tafaloudsutuknoint $= 3{\cdot}14159265358979 3$.
Grey's ingenious device consists mainly in turning numbers into words. The key is:

b	d	t	f	l	s	p	k	n	z
1	2	3	4	5	6	7	8	9	0
a	e	i	o	u	au	oi	ei	ou	y

The number 1851 is akla, or beila, or akub. With this device he combines most skilfully *cadence* and *contraction*.

two right angles; twice tafalout (2π) is the number of inches (to less than one millionth) in the circumference of our circle with radius unity which we may call the *scale-circle*, and a right angle is $\frac{1}{2}\pi = 1{\cdot}570796$ inches measured on this scale-circle. Whenever an angle or arc θ, or a, is spoken of, θ or a, unless the contrary is expressly stated, is *so many inches*, that is, *so many radii* of the circumference of our scale-circle, *and of no other*, measured in the direction $ADBE$, or $AEBD$, according to the sign of θ or a. Here θ or a is a number, not a vague mark of position like A, B, C, (Ang, Bang, Cang.)

Def. 1. The *sine* of any number θ, written $\sin\theta$, is the number of units (i.e. on our scale of inches) in the ordinate y of the extremity (P) on the moveable radius of the arc θ of the scale-circle, $x^2+y^2=1$, about the origin of coordinates: and the *cosine* of the same θ, $\cos\theta$, is the length of the abscissa x (in inches) of the same point (P). Neither of the numbers $\sin\theta$, $\cos\theta$, can exceed unity the radius of the scale-circle, whatever number positive or negative θ may be.

Def. 2. The *tangent* of any number θ, $\tan\theta$, is the fraction or ratio, $\sin\theta:\cos\theta$.

Def. 3. The *cotangent* of θ is the reciprocal of the *tangent*, i.e.

$$\cot\theta = 1:\tan\theta = \cos\theta:\sin\theta.$$

Def. 4. The *secant* of θ is the reciprocal of the *cosine*, and the *cosecant* is the reciprocal of the *sine*, i.e.

$$\sec\theta = 1:\cos\theta, \quad \operatorname{cosec}\theta = 1:\sin\theta.$$

Def. 5. The *versed sine* of θ, $\operatorname{versin}\theta$, or $\operatorname{ver}\theta$, is $= 1 - \cos\theta$.

Thus if θ be the arc AP, and Pp the ordinate of P,

$$Pp = y = \sin\theta, \text{ and } OP = x = \cos\theta.$$

If $AP_1 = -AP$ or $\theta_1 = -\theta$, we see that

$$\sin(-\theta) = -\sin\theta, \text{ for } P_1p = -Pp; \text{ and } \cos(-\theta) = +\cos\theta. \quad \text{(E)}$$

As P is the same point of the circle, and has the same x and y, for the arc θ as for $2\pi+\theta$, a whole revolution $ADBEA$ and θ inches more; or for $(-2\pi+\theta)$, a whole *negative* revolution $AEBDA$ and then θ inches *positive*; the arcs θ and $(\pm 2\pi+\theta)$ have the same sine and cosine; and so have

θ and $(\pm 4\pi + \theta)$, and θ and $(\pm 2n\pi + \theta)$, n being any whole number.

The sine (or cosine) of any number θ is the same number with the sine (or cosine) of the number $(2n\pi + \theta)$, n being any integer, positive or negative. (F)

There are thus an infinity of angles which have the same sine and cosine.

Let AT be the tangent of the scale-circle at A, the point from which angles are measured. By [17, b], OAT is a right angle, and OA, AT, are the x and y of T in the line OP, whose equation is (8) $y{:}x = Pp{:}Op$,

$$\therefore \frac{TA}{OA} = \frac{Pp}{Op} = \frac{\sin\theta}{\cos\theta}; \text{ or } OA \text{ being} = 1,\ TA = \tan\theta,$$

by definition of the tangent above given.

Because of the parallels Pp and TA, we have by [6]

$$\frac{TO}{AO} = \frac{PO}{Op}, \text{ or since } PO = AO = 1,$$

$$TO = \frac{1}{Op} = \frac{1}{\cos\theta} = \sec\theta, \text{ by definition 4.}$$

Let OD be perpendicular to OA, and let DR be the tangent of the curve at D; then is ODR [17] a right angle. If Pq be parallel to OA, Pq and DR, being both at right angles to OD, are parallels; whence by [6],

$$\frac{RO}{DO} = \frac{OP}{Oq}, \text{ or since } Oq = Pp \text{ by [2],}$$

$$RO = \frac{1}{Pp} = \frac{1}{\sin\theta} = \operatorname{cosec}\theta, \text{ by definition.}$$

$$\text{Again, } \frac{DR}{Pq} = \frac{DO}{Oq} \text{ by [6]; whence mult. by } Pq{:}DO,$$

$$\frac{DR}{DO} = \frac{Pq}{Oq}, \text{ or since } Pq = Op \text{ by [2],}$$

$$DR = \frac{Op}{Pp} = \frac{\cos\theta}{\sin\theta} = \cot\theta, \text{ by definition 3.}$$

Also, $Ap = 1 - \cos\theta = \text{versin}\,\theta$, by definition 5.

Let S be *any* point of the line OP, and Ss the ordinate of S. Because Ss and Pp are parallel, we have by [6],

(A) $\dfrac{Os}{OS} = \dfrac{Op}{OP} = Op = \cos\theta = \cos SOs$, and by [6] also

$\dfrac{Ss}{Pp} = \dfrac{SO}{PO}$, or multiplying by $\dfrac{Pp}{SO}$,

(B) $\dfrac{Ss}{SO} = \dfrac{Pp}{PO} = Pp = \sin\theta = \sin SOs.$

Since these are true for any position of P between A and D, they will be true if the triangle SOs be made to stand on the base Ss in the place of Os, and the angle OSs be put in the place of the angle SOs or θ; that is, we should prove inevitably that

(C) $\dfrac{Ss}{OS} = \cos(OSs)$, and $\dfrac{Os}{OS} = \sin(OSs)$ (D).

Def. 6. Two arcs or angles whose sum is *a right angle* are said to be *complements* of each other.

Def. 7. Two arcs or angles whose sum is *two right angles* are said to be *supplements* of each other.

By Prop. D, the three angles of any triangle are equal to two right angles; whence it follows that (SOs) and (OSs) are *complements.*

We have then proved in (A, B, C, D), since SOs may be *any* right-angled triangle, *that the sine of any acute angle is the cosine of its complement*, and *the cosine of any acute angle is the sine of its complement*, i. e.

$$\cos\omega = \sin\left(\frac{\pi}{2} - \omega\right), \quad \text{and} \quad \sin\omega = \cos\left(\frac{\pi}{2} - \omega\right) \qquad \text{G.}$$

From (A) and (C) we see that *in any right-angled triangle, the cosine of an acute angle is the quotient of the adjacent side by the hypothenuse; and that the sine of an acute angle is the quotient of the opposite side by the hypothenuse*, is proved in (B) and (D).

These two properties may be expressed thus. In any right-angled triangle

$$\left.\begin{array}{l}\text{(sine of acute angle) times hypothenuse=opposite} \perp \\ \text{(cosine of acute angle) times hypothenuse=adjacent} \perp\end{array}\right\} \text{(H)}$$

the two sides about the right angle being perpendicular to each other.

By the equation to OP,

$$Ss{:}Os = Pp{:}Op = \sin\theta{:}\cos\theta = \tan\theta\,;\ \text{or,}$$

in any right-angled triangle, the tangent of an acute angle is the quotient of the opposite perpendicular by the adjacent perpendicular. K.

Produce Pq to meet the circle in Q. In the isosceles $\triangle\, POQ$, Oq is biCang by [18], and $\angle\, POq = \angle\, QOq$, wherefore the complements of these are equal, or $\angle\, AOP = \angle\, BOQ$: the latter is the supplement of AOQ (by Prop A.) $\therefore\ AOP$ and AOQ are supplements of each other, and AP or θ, and APQ or $\pi-\theta$, are supplemental arcs. By [2] $Pp = Qq'$, the ordinate y at Q, i.e. $\sin\theta = \sin(\pi-\theta)$; which proves that *the sine of any arc* (AP) *is also the sine of its supplemental arc* (APQ).

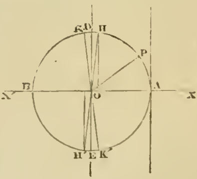
33. As Qq' is the sine, so Oq' is the cosine of APQ; and $Oq' = -Op$ by the equation to the circle $x^2+y^2=1$ giving $x = \pm\sqrt{1-(Pp)^2}$, at P and Q; therefore *the cosine of any arc* (AP) *is equal, but with a contrary sign, to the cosine of its supplemental arc* (APQ).

These properties are, briefly,

$$\sin\theta = \sin(\pi-\theta), \quad \cos\theta = -\cos(\pi-\theta). \qquad \text{(L.)}$$

Hence by (E), $-\sin\theta = \sin(\pi+\theta)$, $-\cos\theta = \cos(\pi+\theta)$. L′.

The variations in sign and magnitude of $\sin\theta$ and $\cos\theta$ are exactly those of y and x in the circle $x^2+y^2=1$, and should be carefully studied.



The sine and cosine of an arc vary *with*, but not *like*, the arc θ, and are said to be, as well as $\tan\theta$, *circular functions of* θ.

As the moveable radius OP revolves, $\cos\theta$ and $\sin\theta$ are continually varying whilst the arc θ is continually increasing; but neither of them can ever exceed unity, the radius of the circle: the tangent and secant vary too continually from $\frac{-1}{0}$ to $\frac{1}{0}$, from negative to positive infinity.

At D, for $\theta = \frac{\pi}{2}$, $\tan\theta = \frac{1}{0}$, because its denominator $\cos\theta = 0$; and $\sec\theta = \frac{1}{0}$: as is evident from the figure; for T, the intersection of the radius OD with AT, is infinitely distant. When P is between D and B, or between A and E, the sine and cosine have different signs, being the y and x of the point P; and their quote is negative, i.e. $\tan\theta$ is negative; when P is between A and D, or between B and E, $\sin\theta$ and $\cos\theta$ have like signs, and their quote, or $\tan\theta$, is positive. The tangent of the arc AH, when H is very near to D, is an exceedingly great positive number, being $\sin\theta : \cos\theta$, the denominator of which, $\cos\theta$, is exceedingly small, (11): and exceedingly great positively at H', when it is $-\sin\theta_1 : -\cos\theta_1$, $\cos\theta_1$ being an indefinitely small fraction.

At K and K', supposed very near to D and E on the opposite sides of DE from H and H', the tangent is an exceedingly great negative number.

k. At A and B, $\sin\theta$ and $\tan\theta$ are zero: they both change sign with θ in passing through zero at A, which is the only point at which θ is zero: and they change sign again in passing through the value 0, when $\theta = \pi$, at B. $\sin\theta$ and $\sin(-\theta)$ have contrary signs, θ being the same in both; thus, $\sin 1{\cdot}1 = -\sin(-1{\cdot}1)$; and generally

$$\sin(\pm\theta_1) = -\sin(\mp\theta_1); \quad \tan(\pm\theta_1) = -\tan(\mp\theta_1),$$

taking upper signs together, or lower together.

l. At D and E, on the axis of y, $\cos\theta = 0$, and consequently $\cot\theta$ is nothing. $\cos\theta$ does not pass through zero with θ, but changes sign when θ passes through $\frac{1}{2}\pi$; i.e. the cosine changes sign in passing through zero at $\theta = \frac{1}{2}\pi$, and the tangent consequently changes sign at the same point of the circle in passing through infinite, as it does again when $\theta = \frac{3}{2}\pi$, passing again through $\frac{1}{0}$.

m. The greatest value of $\sin\theta$ is unity, when $\theta = \frac{1}{2}\pi$, its least is -1, when $\theta = \frac{3}{2}\pi$. The greatest value of $\cos\theta$ is $+1$, when $\theta = 0$, or $= 2n\pi$: the least value is -1, when $\theta = (2n+1)\pi$, n being any whole number. Neither $\sin\theta$ nor $\cos\theta$ changes sign in passing through $+1$, or through -1.

$\sec\theta$, $\operatorname{cosec}\theta$, $\cot\theta$, have of course the signs of their reciprocals $\cos\theta$, $\sin\theta$, and $\tan\theta$.

At every point of the circle, for all finite values, positive or negative, of θ, since $x^2+y^2=1$, we have always

$$\sin^2\theta+\cos^2\theta=1, \qquad \text{M.}$$

You will remember that the number π is by definition the number of inches or unit radii in the semi-circumference of our scale circle, and that nothing is hereby affirmed or implied concerning the exact length of any other semi-circle.

I fancy, Richard, you find some of this rather difficult.

Rich.:—Indeed I do, and have left to Jane the honour of following you all through this lesson. How do you feel, dear sister mine, after this feast of trigonometry?

Jane:—A little fatigued; but more by the variety than the difficulty of the subject.

Rich.:—I am waiting for the mnemonics; when I have mastered them, and all their meaning, I shall know what to think about, and I dare say it will be with me as it has been before; I shall first get a clear notion of what is to be proved, and then obtain the proof by dint of a little thinking and conversation with you.

Uncle Pen.:—I advise you both first to make sure of the mnemonics I am about to give you, and of their exact signification: you will find little trouble in the demonstrations. To be able accurately to lay down a proposition is a considerable step towards the proof of it.

[21] x and y are Cos and Sī
ŏf Scăle-a′rc from OX; cĕn. o′r Ax Rīgh. Def. 1. (32.)

*Cos*ine and *Sine* *of arc* of *scale* circle, measured *from* the line OX; the circle having its *cen*tre at *or*igin, and referred to *righ*t *Ax*es; viz. $x^2+y^2=1$.

[22] So′rCo.Po′th is o′p or ad; (H). vid. poth. [7].
Sī bȳ Cŏ′s ta′n, is vi(ŏp. a′d). Def. 2. (K.) 's for =

SorCo is *S*in *or* *Co*s. *S*in × poth = *op* ⊥ (H), and *Cos* × poth = *ad* ⊥; *op*posite, *ad*jacent, ⊥. *S*in θ by *Cos* θ = *tan* θ = op. ⊥ by ad. ⊥, (K). *vi* for quote.

[23] Co′rSin ω' 's SĭnŏrC(ri′ght min ω') (G.) pron. ω, ō.
Co′rS is le′m the Co′rS of − ple′m (L.)
CorS(π et ω) is le′ CorSω. (L′.) pron. pĕtώ.

CorSin, or CorS, for *Co*sine *or* *S*ine. SinorC. for *Sine* *or* *C*osine.

(right min ω) is (*right* angle *min*us ω); *min* means $-$ under a vinculum; lem is $\mp$ *le*ss or *m*ore; $\text{Cos}\,\theta = -\text{Cos}\,(\pi - \theta)$, $\text{Sin}\,\theta = \sin\,(\pi - \theta)$; $\pi - \theta$ is sup*plem*ent of θ; (π et ω) a dissyll.; et is $+$ under a vinculum. $\text{Cos}\,(\pi + \omega) = -\cos\omega$; $\sin\,(\pi + \omega) = -\sin\omega$; le is $-$.

[24] Tăn Co's Sĭn are rec. (pron. reck.)
Cŏtă Se'c Cŏse'c. (Def. 3, 4.)

$\text{Tan}\,\theta$, $\text{Cos}\,\theta$, $\text{Sin}\,\theta$, are in order the *rec*iprocals of *Cotan* θ, $\text{Sec}\,\theta$, $\text{Cosec}\,\theta$. Two *or*'s in a line correspond, as to antecedents and consequents, in [23], [22].

DUQ(Sĭ Cŏ)'s One: (M) DUQ, vid. [13.]
[25] Co right is none
Co none is one.

i.e. $\sin^2\theta + \cos^2\theta = 1$; Cos (right angle) $= 0$; $\text{Cos}(0) = 1$.

Jane:—The lesson is now brought into a manageable compass, by these five little mnemonics: but what queer numbers these circular functions, as you call them, are! When I have a number θ given, I can understand what is meant by its half or its square, or its cube root, and I can find any of these by an arithmetical operation; there is a rule for it. But what sort of a sum in arithmetic would you do to find the sine of a number? What kind of a rule for finding a sine can be laid down, such that the answer shall never be greater than unity, whatever number θ you work with, and that thousands of different numbers θ shall give the same proper fraction for the answer?

Uncle Pen.:—At present it is enough for you to know what has been proved, and to be able to find, in the tables of Hutton or others, the value of $\sin\theta$ for every number θ you may choose. You shall have complete and mnemonical replies to your questions, and to many others equally curious, when we come to that part of our subject.

Ex. If the hypothenuse is 100, the side a 60, and $b = 80$,

$$\text{Sin}\,A = \tfrac{3}{5},\quad \text{Cos}\,A = \tfrac{4}{5},\quad \text{Tan}\,A = \tfrac{3}{4},$$

$$\text{Vers.}\,A = \tfrac{1}{5},\quad \text{Sec}\,A = \tfrac{5}{4},\quad \text{Cosec}\,A = \tfrac{5}{3},\quad \text{Cot}\,A = \tfrac{4}{3},$$

$$\text{Sin}\,B = \tfrac{4}{5},\quad \text{Cos}\,B = \tfrac{3}{5},\ \&c.$$

The external obtuse angles are $\pi - A$, $\pi - B$, $\text{Sin}\,(\pi - A) = \tfrac{3}{5}$, $\text{Cos}\,(\pi - A) = -\tfrac{4}{5}$, $\text{Tan}\,(\pi - A) = -\tfrac{3}{4}$.

LESSON VIII.

34. Let ABC be any triangle; let the sides opposite the angles A, B, C, be a, b, c;

let $p = CD$, be the perpendicular from C on c;

... $s = BD$, be the segment of c between D and B,

and $(c \pm s) = AD$, ... D and A;

the lower sign being taken when B is an acute, and the upper when B is an obtuse angle; for when B is obtuse, $AD > AB$, otherwise the $\triangle BCD$ would contain an obtuse and a right angle, or more than two right angles, which is impossible by Prop. D. By [7],

$$b^2 = p^2 + (c \pm s)^2,$$

$$a^2 = p^2 + s^2; \text{ whence by subtraction,}$$

$$b^2 - a^2 = (c \pm s)^2 - s^2 = \text{by [14, a, b] } c^2 \pm 2cs + s^2 - s^2, \text{ or}$$

$$b^2 = a^2 + c^2 \pm 2cs = a^2 + c^2 \pm s \cdot 2c. \qquad \text{(A)}$$

When B is acute, we take the lower sign, or

$$b^2 = a^2 + c^2 - 2cs. \qquad \text{(a)}$$

Now in the right-angled triangle DBC, Cos (DBC) $\times$ $CB = DB$ by [22], (Cos $\times$ poth = ad. $\perp$) i.e., since ABC is in this case DBC,

$$a \times \text{Cos}\,(ABC) = s = a \,\text{Cos}\, B;$$

wherefore the above becomes, putting a Cos B for s,

$$b^2 = a^2 + c^2 - 2ac \,\text{Cos}\, B. \qquad \text{(B)}$$

We may consider AB and C to be *numbers*, and not *merely* symbols of *position*.

When B is obtuse we have

$$b^2 = a^2 + c^2 + 2cs. \qquad \text{(a')}$$

As before, Cos $(DBC) \times CB = DB$; but (Def. 7, 33) DBC is the supplement of ABC in this case; so that [23, L] Cos $DBC = -$ Cos ABC.

$$\text{Therefore } -\text{Cos}(ABC) \times CB = DB, \text{ or}$$

$$-\text{Cos}\, B \times a = s,$$

by which (a′) becomes

$$b^2 = a^2 + c^2 - 2ac\, \text{Cos}\, B, \qquad \text{(B)}$$

exactly as before when we considered B acute, so that (B) is true for *any* angle B of any triangle; b being always the side opposite B.

We may then *go round the circle, bcabc*, vid. (17), and write

$$\left.\begin{aligned} c^2 &= b^2 + a^2 - 2ba\, \text{Cos}\, C, \\ a^2 &= c^2 + b^2 - 2cb\, \text{Cos}\, A, \end{aligned}\right\} \qquad \text{(B)}$$

In any triangle, the square of any side is obtained by subtracting from the sum of the squares of the other two sides, twice the product of those two sides multiplied by the Cosine of the angle of the triangle contained by them.

Obs. This subtraction is in fact an *addition*, if the cosine be *negative.*

When the contained angle is a right angle, its Cosine vanishes, and this becomes [7] the theorem of Pythagoras; but you are not to consider this to be our *proof* of that theorem, although the result (B) includes [7] as a particular case; for we have employed [7] continually as a premiss in the reasoning which establishes (B).

The Proposition (A), expressed in words at length, is none other than the 12th and 13th propositions of the second book of Euclid. By (B) we find any side of a triangle if the other two sides and the angle between them be given; for the tables (of whose use more hereafter) give at a glance the Cosine of the given contained angle, and the sought side is the square root of the known number on the right side of (B). We see from (B) also that if two triangles have two sides $a = a_1$, $c = c_1$, and the angle B between a and c, equal to that between a_1 and c_1; they have their third sides equal, $b = b_1$; a truth which has been established in (20) and (30) by the simpler argument that such a pair of triangles will cover each other. Hence

The diagonal of a parallelogram bisects it. D.

These results are most important, and *must* be mastered. I will give you a mnemonic both for (A) (B) and for the proof of them.

Draw pérc: SUD (bá) is SŬD (ségs of c′).
And sq′.b is DUQ ác mol (ség. op) twō c,
[26] As ’tùse or ’cùte is Báng op. b; (A)
Or, sq′.b is DUQ ác le CoBáng·two ác. (B)

Draw *perc* or CD, or, let CD *be supposed* to be $\perp$ to c, [18]; then we prove $b^2 - a^2 = (AD)^2 - (BD)^2$, vid. SŬD [20]; *segs* for *segments*; sq.b pron. squib; DUQ[13]; $\pm$ = mol; *seg. op*. is the *seg*ment of c *op*posite, not adjacent to b, whose square we are expressing. Pron. twō long, to avoid confounding it with *to*; + or − in *mol*, with B ob*tuse* or a*cute*; B = Bang opposite b, [18]; *le* is −; *CoBang two ac* means, of course, Cos B times $2ac$, $\times$ being understood between quantities joined together, as also in $\pm$ $s2c$, (A), in the second line.

From (B) we deduce, by transposition and division,

(C) $$\text{Cos}\, B = \frac{a^2 + c^2 - b^2}{2ac}; \quad \text{Cos}\, C = \frac{b^2 + a^2 - c^2}{2ba};$$

$$\text{Cos}\, A = \frac{c^2 + b^2 - a^2}{2cb};$$

by which we obtain all the angles of a triangle from knowing its three sides. Thus the triangle whose sides are

$$a = 6, \quad b = 8, \quad c = 9, \text{ gives}$$

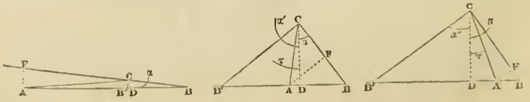
$$\text{Cos}\, B = \frac{36+81-64}{108}; \quad \text{Cos}\, C = \frac{64+36-81}{96};$$

$$\text{Cos}\, A = \frac{81+64-36}{144};$$

$$\text{or } \text{Cos}\, B = \frac{53}{108}, \quad \text{Cos}\, C = \frac{19}{96}, \quad \text{Cos}\, A = \frac{109}{144}.$$

The tables give the angles *less than two right angles*, which have these cosines, and *these* are evidently the angles of the triangle.

35. Let ABC be any triangle, and let CD be *supposed* perpendicular from C upon the side c (or AB), and AF



perpendicular from A in the side a (or BC). Then $c.(CD) = a(AF)$ must be true, c and a being the lengths of the sides opposite C and A, and (CD) and (AF) representing the lengths of the perpendiculars on them; for either of these products is twice the area of the $\triangle\, ABC$, by Prop. B, (20). Let the angle

$$DCB = \alpha,\ \ DCA = \tau;\ \text{then as } AF = b \text{ Sin } C,\ [22]$$

$$c.(CD) = ab.\text{Sin}(\alpha \pm \tau),\ \text{i.e.} = ab \text{ Sin } C,$$

taking the upper or lower sign as CD falls within or without ABC. Or

$$(DB \pm DA)(CD) = ab \text{ Sin}(\alpha \pm \tau);\ \text{or}$$

$$DB.CD \pm DA.CD = ab.\text{Sin}(\alpha \pm \tau);$$

but [22]

$$DB = a \text{ Sin } \alpha,\quad CD = b \text{ Cos } \tau = a \text{ Cos } \alpha,\quad DA = b \text{ Sin } \tau;$$

$$\therefore\ a.\text{Sin } \alpha.b.\text{Cos } \tau \pm a.\text{Cos } \alpha.b.\text{Sin } \tau = ab.\text{Sin}(\alpha \pm \tau),$$

or, div. by ab, (vide *The Mathematician*, Vol. II. p. 134)

$$\text{Sin } \alpha.\text{Cos } \tau. \pm \text{Cos } \alpha.\text{Sin } \tau. = \text{Sin}(\alpha \pm \tau). \qquad \text{(a)}$$

The *upper* of these signs is to be taken when α and τ are *any* angles between *perc* falling *within any* triangle and the sides a, b; the *lower* when α and τ are angles between *perc* falling *without*, and the sides a, b.

Let CB' be drawn $\perp$ to CB, so that BCB' may be a right angle. Then, if $\angle\, \alpha' = B'CD$, it is evident that $\angle\, B'CA = (\alpha' \pm \tau)$, + or −, according as the $\perp\, CD$ falls within or without the $\triangle\, AB'C$: and, by the same reasoning with the above, we have

$$\text{Sin } \alpha' \text{ Cos } \tau \pm \text{Cos } \alpha' \text{ Sin } \tau = \text{Sin}(\alpha' \pm \tau).$$

Now $\alpha' = \frac{\pi}{2} - \alpha$, $\therefore$ Sin α' = Cos α, and Cos α' = Sin α, by [23, G]

$$\text{and Sin}(\alpha' \pm \tau) = \text{Sin}\left(\frac{\pi}{2} - \alpha \pm \tau\right) = \text{Sin}\left\{\frac{\pi}{2} - (\alpha \mp \tau)\right\}\ [11]$$

$$= \text{Cos}(\alpha \mp \tau),\ \text{by } [23, G]$$

$$\therefore\ \text{Cos } \alpha \text{ Cos } \tau \pm \text{Sin } \alpha \text{ Sin } \tau = \text{Cos}(\alpha \mp \tau),$$

which is the same with

$$\text{Cos } \alpha.\text{Cos } \tau \mp \text{Sin } \alpha.\text{Sin } \tau = \text{Cos}(\alpha \pm \tau), \qquad \text{(b)}$$

the upper signs taken together, or the lower together.

We have thus found the expansion of a sine or cosine of a sum or difference of two numbers a and τ, in terms of the sines and cosines of those numbers, when neither of them is greater than $\frac{\pi}{2}$; for our demonstration does not apply to the case of the angles a or $\tau =$ a right angle, because *perc* (CD) cannot be at right angles to *two* sides of the triangle. It has yet to be proved that (a) and (b) are true for arcs a and τ equal to and greater than a quadrant $\frac{\pi}{2}$. The cases to be considered are,

1. $a = \frac{\pi}{2}$,
2. $\tau = \frac{\pi}{2}$,
3. $a = \tau = \frac{\pi}{2}$
4. $a > \frac{\pi}{2} < \pi$,
5. $\tau > \frac{\pi}{2} < \pi$,
6. both $> \frac{\pi}{2} < \pi$.
7. $a > \pi$.
8. $\tau > \pi$,
9. both $> \pi$,
10. $a > \pi$, $\tau > \frac{\pi}{2} < \pi$,
11. $\tau > \pi$, $a > \frac{\pi}{2} < \pi$,
12. $a > \frac{3\pi}{2} < 2\pi$,
13. $\tau > \frac{3\pi}{2} < 2\pi$,
14. both $> \frac{3\pi}{2} < 2\pi$.

The cases of a, or τ, or both, greater than 2π or four quadrants, may be neglected; for if $a = (2n\pi + \theta)$ and $\tau = (2n'\pi + \theta')$, a, τ, and $(a \pm \tau)$ have the sines and cosines of θ, θ', and $(\theta \pm \theta')$ by (F, 32). The equations (a) and (b) are perfectly general, for all values, short of $\pm$ infinite, of both a and τ. You will find it a most profitable and by no means difficult exercise, to verify them for all the above cases, and for a and τ of either sign: all that you have to do is, if you suppose e.g. $a > \pi$, to put for a, on both sides of the equation, $(\pi + a_1)$, a_1 being supposed less than a quadrant; and then to shew that each of the equations (a), (b) remains true after the substitution for either upper or lower signs taken together, by some of the following considerations.

The arcs $\left(\frac{\pi}{2} \pm \theta\right)$ and $(\mp \theta)$ are complements, and $(\pi \pm \theta)$

and $(\mp\theta)$ are supplements, as are also $\left(\frac{\pi}{2} \pm \theta\right)$ and $\left(\frac{\pi}{2} \mp \theta\right)$, whatever θ may be, if in these pairs of arcs the upper signs or the lower be taken together.

Hence $\text{Cos}\left(\frac{\pi}{2} \pm \theta\right) = \text{Sin}\,(\mp\theta)$, &c. [23, G];

$$\text{Sin}\,(\pi \pm \theta) = \text{Sin}\,(\mp\theta)\ [23, \text{L}] = \mp \text{Sin}\,\theta,\ (32, \text{k});$$

whence also $\text{Sin}\,(\alpha - \frac{1}{2}\pi) = -\ \text{Sin}\,(\frac{1}{2}\pi - \alpha)$.

Remember that if $\pm\theta$ terminates at (x_1, y_1), $\pm(\pi + \theta)$ terminates at $(-x_1, -y_1)$ on the same diameter of the circle.

As an example I take case 10;

$$\text{let } \alpha = \pi + \alpha_1, \quad \tau = \tfrac{1}{2}\pi + \tau_1, \quad \alpha_1 \text{ and } \tau_1 \text{ each} < \tfrac{1}{2}\pi.$$

Then (a) is

$$\text{Sin}\,(\pi + \alpha_1)\,.\,\text{Cos}\,(\tfrac{1}{2}\pi + \tau_1) \pm \text{Sin}\,(\tfrac{1}{2}\pi + \tau_1)\,.\,\text{Cos}\,(\pi + \alpha_1)$$
$$= \text{Sin}\,\{\pi + \alpha_1 \pm (\tfrac{1}{2}\pi + \tau_1)\}. \quad (\text{a}')$$

Now in the first member of this,

$$\text{Sin}\,(\pi + \alpha_1) = \text{Sin}\,(-\alpha_1)\ [23, \text{L}] = -\ \text{Sin}\,\alpha_1:$$
$$\text{Cos}\,(\tfrac{1}{2}\pi + \tau_1) = \text{Sin}\,(-\tau_1)\ [23, \text{G}] = -\ \text{Sin}\,\tau_1,$$
$$\text{Sin}\,(\tfrac{1}{2}\pi + \tau_1) = \text{Cos}\,(-\tau_1) \qquad = \ \text{Cot}\,\tau_1,\ (33, \text{m})$$
$$\text{Cos}\,(\pi + \alpha_1) \qquad = -\ \text{Cos}\,\alpha_1,\ [23, \text{L}]$$

Therefore (a′) becomes

$$-\ \text{Sin}\,\alpha_1 \times -\ \text{Sin}\,\tau_1 \pm (\text{Cos}\,\tau_1 \times -\ \text{Cos}\,\alpha_1)$$
$$= \text{Sin}\,\{\pi + \alpha_1 \pm (\tfrac{1}{2}\pi + \tau_1)\}. \qquad (\text{a}')$$

Taking the upper sign in the second member,

$$\text{Sin}\,(\pi + \alpha_1 + \tfrac{1}{2}\pi + \tau_1) = [23, \text{L}'] - \text{Sin}\,(\alpha_1 + \tau_1 + \tfrac{1}{2}\pi)$$
$$= -\ \text{Cos}\,(-\alpha_1 - \tau_1) = -\ \text{Cos}\,(\alpha_1 + \tau_1).$$

Taking the lower sign,

$$\text{Sin}\,(\pi + \alpha_1 - \tfrac{1}{2}\pi - \tau_1) = \text{Cos}\,(\tau_1 - \alpha_1) = \text{Cos}\,(\alpha_1 - \tau_1)\ (33, \text{m}).$$

Hence, (a′) is reduced to the two equations, attending to [3] 'like signs,'

$$\text{Sin}\,\alpha_1\,.\,\text{Sin}\,\tau_1 - \text{Cos}\,\alpha_1\ \text{Cos}\,\tau_1 = -\ \text{Cos}\,(\alpha_1 + \tau_1),$$
$$\text{Sin}\,\alpha_1\,.\,\text{Sin}\,\tau_1 + \text{Cos}\,\alpha_1\ \text{Cos}\,\tau_1 = \ \text{Cos}\,(\alpha_1 - \tau_1),$$

both which have been proved true for all values of α, τ, $< \frac{1}{2}\pi$.

The above looks perplexing, but there is here nothing to be committed to memory beyond the propositions (a) and (b), and the reasoning by which they are established. A long multiplication sum is perplexing; but you are qualified to cope with it, if you know and can handle your multiplication table. You cannot *see* the correctness of a product of large factors, as you can that of $2 \times 3 = 6$; but you are satisfied if the *work* step by step is right. Be then equally satisfied with the results of our symbolical arithmetic, although you cannot *see* your way at a glance from the premises to the conclusion. It is enough, if you are able to *find* it.

I give you now a mnemonic for (a) and (b), and another for their proof, in case you should find any difficulty in remembering the steps of the latter. Pronounce α and τ like *a* and *t*.

[27] (*a*) Sĭn ($\acute{\alpha}$ mŏl τ) ǐs S$\acute{\alpha}$.Coτ mŏl C$\acute{\alpha}$.Siτ; mol τ, one syl.

(*b*) Cŏs ($\acute{\alpha}$ mol τ) is C$\acute{\alpha}$.Coτ lĕm S$\acute{\alpha}$.Siτ.

mol, v. [14]; lem, v. [23].

c pérc is *ba* Sĭ Cáng;

then pút (α mól τ) for Cang;

(new—óld *a*) máke right áng.

The three last lines contain the steps of the demonstration. Our first step was $c.CD = ba\text{Sin}C$; CD is $\perp$ on c, vid. [18]: the next is to put for C the sum or difference of angles made by CD, $(\alpha \pm \tau)$: then we applied [22], and obtained (*a*): we then drew a *new a*, CB' *mak*ing BCB' a *right ang*le between *old a* and *new a*, and obtained (*b*).

LESSON IX.

36. *Richard:*—How I long for an amusing question! The last lesson was so dry. I saw the surveyor this morning measuring Holt's farm; and I wondered how he could find the number of square acres and make a plan of it, by measuring a few lines here and there.

Jane:—If there had been any triangular fields, I fancy you would have a perpendicular drawn on a large scale by the theorem of Pythagoras, to get the product of the base and altitude.

Uncle Pen.:—Most likely not: the surveyor finds it more convenient to form a plan by the aid of a few leading lines, and then draws his perpendiculars on the paper. Neither is it necessary to draw a perpendicular at all, to find the area of a triangle. We will solve presently the two questions following.

Two fields contain together $11\frac{3}{4}$ *acres, and one of them contains* $3\frac{5}{7}$ *acres more than the other. What is the area of each field?*

A triangular field has its sides 600, 560, *and* 720 *feet in length; what is the area of the field?*

You will be able to solve these yourselves, if you will pay attention to what I have further briefly to say about [27].

$$\text{Sin}\,(\alpha+\tau)=\text{Sin}\,\alpha\,.\,\text{Cos}\,\tau+\text{Cos}\,\alpha\,.\,\text{Sin}\,\tau\ldots\ldots(1),$$

$$\text{Sin}\,(\alpha-\tau)=\text{Sin}\,\alpha\,.\,\text{Cos}\,\tau-\text{Cos}\,\alpha\,.\,\text{Sin}\,\tau\ldots\ldots(2),$$

$$\text{Cos}\,(\alpha+\tau)=\text{Cos}\,\alpha\,.\,\text{Cos}\,\tau-\text{Sin}\,\alpha\,.\,\text{Sin}\,\tau\ldots\ldots(3),$$

$$\text{Cos}\,(\alpha-\tau)=\text{Cos}\,\alpha\,.\,\text{Cos}\,\tau+\text{Sin}\,\alpha\,.\,\text{Sin}\,\tau\ldots\ldots(4).$$

These four equations are true for any values of α and τ: provided that α and τ mean the same two numbers on both sides of the same equation.

Let α as well as τ be the same number in the equations 1 and 2; then by addition $(1+2)$, and by subtraction $(1-2)$,

$$\text{Sin}\,(\alpha+\tau)+\text{Sin}\,(\alpha-\tau)=2\,\text{Sin}\,\alpha\,.\,\text{Cos}\,\tau\ldots\ldots(5),$$

$$\text{Sin}\,(\alpha+\tau)-\text{Sin}\,(\alpha-\tau)=2\,\text{Cos}\,\alpha\,.\,\text{Sin}\,\tau\ldots\ldots(6).$$

In like manner from 3 and 4, $(4+3)$ and $(4-3)$ give

$$\text{Cos}\,(\alpha-\tau)+\text{Cos}\,(\alpha+\tau)=2\,\text{Cos}\,\alpha\,.\,\text{Cos}\,\tau\ldots\ldots(7),$$

$$\text{Cos}\,(\alpha-\tau)-\text{Cos}\,(\alpha+\tau)=2\,\text{Sin}\,\alpha\,.\,\text{Sin}\,\tau\ldots\ldots(8).$$

The following elegant little transformation will bring you to the solution of our first question. Because $b-b=0$,

$$2a=a+a+b-b=(a+b)+(a-b),$$

$$2b=b+b+a-a=(a+b)-(a-b),\ \text{v. [11]},$$

$$\therefore\ a=\frac{a+b}{2}+\frac{a-b}{2},\ \text{and}\ b=\frac{a+b}{2}-\frac{a-b}{2}\ldots\ldots(9);$$

put $(a+b)=\text{SuM}$, $(a-b)=\text{Diff.}$

(9) is, $\frac{1}{2}\,\text{S}\pm\frac{1}{2}\,\text{D}=a$, or b, as you take $+$ or $-$.

[28] HaS (ab) mol HaD (ab) is a or b;

i.e., *ha*lf the *S*um of (a and b) + *ha*lf the *D*iff. of (a and b) = a,
ha ...... *S*..................... − *ha* ...... *D*..................... = b,
whatever two numbers a and b may be; a, the *first* named, being *usually* not *necessarily* supposed the *greater*.

If then A, B, be the areas of the two fields,

$$A = \tfrac{1}{2} \times 11\tfrac{3}{4} + \tfrac{1}{2} \times 3\tfrac{5}{7}, \text{ and } B = \tfrac{1}{2} \text{ of } 11\tfrac{3}{4} - \tfrac{1}{2} \text{ of } 3\tfrac{5}{7},$$

giving A and B in acres. You may reduce these to a more workmanlike shape yourselves.

Put now

$\alpha + \tau = S$, $\alpha - \tau = D$; $\alpha = \frac{1}{2}(S + D)$, $\tau = \frac{1}{2}(S - D)$ by [28]: the equations 5, 6, 7, 8, become

$$\text{Sin } S + \text{Sin } D = 2 \text{ Sin } \tfrac{1}{2}(S + D) . \text{Cos } \tfrac{1}{2}(S - D) \ldots\ldots (10),$$
$$\text{Sin } S - \text{Sin } D = 2 \text{ Cos } \tfrac{1}{2}(S + D) . \text{Sin } \tfrac{1}{2}(S - D) \ldots\ldots (11),$$
$$\text{Cos } S + \text{Cos } D = 2 \text{ Cos } \tfrac{1}{2}(S + D) . \text{Cos } \tfrac{1}{2}(S - D) \ldots\ldots (12),$$
$$\text{Cos } D - \text{Cos } S = 2 \text{ Sin } \tfrac{1}{2}(S + D) . \text{Sin } \tfrac{1}{2}(S - D) \ldots\ldots (13).$$

Now S and D may be *any* numbers of either sign, for α and τ, whose values are determined [28] by given values of S and D, are *general* symbols; we can therefore write α for S and τ for D; for either S and D, or α and τ, stand for any pair of numbers you please:

$$\text{Sin } \alpha \pm \text{Sin } \tau = 2 \begin{matrix}\text{Sin}\\ \text{Cos}\end{matrix}\left\{\tfrac{1}{2}(\alpha + \tau)\right\} \times \begin{matrix}\text{Cos}\\ \text{Sin}\end{matrix}\left\{\tfrac{1}{2}(\alpha - \tau)\right\}. \ (10,\ 11).$$

$$\text{Cos } \alpha \pm \text{Cos } \tau = \pm 2 \begin{matrix}\text{Cos}\\ \text{Sin}\end{matrix}\left\{\tfrac{1}{2}(\alpha + \tau)\right\} \times \begin{matrix}\text{Cos}\\ \text{Sin}\end{matrix}\left\{\tfrac{1}{2}(\alpha - \tau)\right\},\ (12,\ 13).$$

Suppose $\alpha = \tau$ in (1) and (3), we have

$$\left.\begin{aligned}\text{Sin } 2\alpha &= 2 \text{ Sin } \alpha \text{ Cos } \alpha,\\ \text{Cos } 2\alpha &= (\text{Cos } \alpha)^2 - (\text{Sin } \alpha)^2,\end{aligned}\right\} \ldots\ldots\ldots (16)$$

or since α is *any* number, so that it have the same meaning on both sides,

$$\left.\begin{aligned}\text{Cos } \alpha &= \text{Cos}^2\left(\tfrac{1}{2}\alpha\right) - \text{Sin}^2\left(\tfrac{1}{2}\alpha\right),\\ \text{Sin } \alpha &= 2 \text{ Cos}\left(\tfrac{1}{2}\alpha\right) . \text{Sin}\left(\tfrac{1}{2}\alpha\right),\end{aligned}\right\} \ldots\ldots\ldots (16)$$

Observe that in (3), $\text{Cos } \alpha . \text{Cos } \tau$ does not become $\text{Cos}^2\alpha^2$, when $\alpha = \tau$, but simply $\text{Cos } \alpha \times \text{Cos } \alpha = \text{Cos}^2\alpha$, the squared cosine of α: $\text{Cos}^2\alpha^2$ would be the squared cosine of α^2, a different arc.

Our second problem can now be solved in a trice; but first let me give you mnemonics for 5, 6, 7, 8, 10...16. The first four are condensed thus,

$$\text{Sin}\,(\alpha+\tau) \pm \text{Sin}\,(\alpha-\tau) = 2\,\frac{\text{Sin}}{\text{Cos}}\,(\alpha)\,\frac{\text{Cos}}{\text{Sin}}\,(\tau), \text{........}(5,\ 6)$$

$$\text{Cos}\,(\alpha-\tau) \pm \text{Cos}\,(\alpha+\tau) = 2\,\frac{\text{Cos}}{\text{Sin}}\,(\alpha)\,\frac{\text{Cos}}{\text{Sin}}\,(\tau), \text{........}(7,\ 8)$$

or

Sin M (+ or −) Sin $D = 2$ (Sin or Cos) α (Cos or Sin) τ, (5, 6)
Cos D (+ or −) Cos $M = 2$ (Cos or Sin) α (Cos or Sin) τ, (7, 8)
which may be pronounced, omitting α and τ, thus:

[29] SíM mol SíD's two SórC.CorS, (5, 6) vid. M [15], D. [9].
CóD mol CóM's two CórS.CorS. (7, 8) 's for =.

Taking antecedents of the or's with +, and consequents with −; Sin (or cos) times cos (or sin) is SorC. CorS, a dissyl.

[30] S$\acute{\alpha}$ mol Síτ's two SórCHaM.CórSHaD, ...(10, 11)
C$\acute{\alpha}$ mŏl Cŏτ's mól twŏ CŏrSHáM.CŏrSHáD... (12, 13)

*S*ine (*or C*os) *H*alf Su*M* × *C*os (*or S*in) *H*alf *D*iff. in (10, 11.) M and D are the suM and Diff. of α and τ on the left. Pron. CH and SH as in Cheshire.

[31] CórS is SU'D or twó CHa.SH. (16) v. SUD [20].

*C*os *or S*in is SUD (*C*os *H*alf, *S*in *H*alf) or two (*C*os *H*alf. *S*in *H*alf.) You should frequently write out these expressions (5, 6,) (10, 11,) &c. from memory, talking to yourself in these mnemonic syllables.

Let the sides of our field be a, b, c: by [26] 'sq. b, &c.'

$b^2 = a^2 + c^2 - 2ac$ Cos B, and by [31]

Cos B = Cos$^2 \frac{1}{2} B$ − Sin$^2 \frac{1}{2} B$; 'Cos is SuD CHaSH.'

but $0 = 1 -$ Cos$^2 \frac{1}{2} B$ − Sin$^2 \frac{1}{2} B$, by [25] 'DUQ SiCo.'

∴ Cos $B = 2$ Cos$^2 \frac{1}{2} B - 1$, by subtraction. P.

Hence $b^2 = a^2 + c^2 - 2ac\,(2\,\text{Cos}^2 \tfrac{1}{2} B - 1)$,

$b^2 = a^2 + c^2 + 2ac - 4ac\,\text{Cos}^2 \tfrac{1}{2} B$, a.

$b^2 = (a+c)^2 - 4ac\,\text{Cos}^2 \tfrac{1}{2} B$, by [14];

then transposing, and dividing by 4,

$$ac\,.\,\text{Cos}^2 \tfrac{1}{2} B = \frac{(a+c)^2 - b^2}{4} = \frac{(a+c)+b}{2} \cdot \frac{(a+c)-b}{2},$$

by [14, c]. A.

here $(a+c)^2$ is the 'sq.b,' and b^2 is the 'sq.a' of [14, c].

Again by *addition* instead of *subtraction* above,

$$\text{Cos } B = 1 - 2\ \text{Sin}^2 \tfrac{1}{2} B. \qquad \text{Q.}$$

Hence $b^2 = a^2 + c^2 - 2ac\,(1 - 2\ \text{Sin}^2 \tfrac{1}{2} B)$,

$$b^2 = a^2 + c^2 - 2ac + 4ac\ \text{Sin}^2 \tfrac{1}{2} B$$

$$= (a-c)^2 + 4ac\ \text{Sin}^2 \tfrac{1}{2} B, \qquad \text{b.}$$

$$ac\ \text{Sin}^2 \tfrac{1}{2} B = \frac{b^2 - (a-c)^2}{4} = \frac{b-(a-c)}{2} \cdot \frac{b+(a-c)}{2}, \text{ by [14, c]}$$

$$= \frac{b-a+c}{2} \cdot \frac{b+a-c}{2}, \text{ by [11]}. \qquad \text{B.}$$

Next by multiplication of equals by equals in (A) and (B),

$$a^2c^2 .\ \text{Cos}^2 \tfrac{1}{2} B\ \text{Sin}^2 \tfrac{1}{2} B = \frac{a+b+c}{2} \cdot \frac{a-b+c}{2} \cdot \frac{a+b-c}{2} \cdot \frac{-a+b+c}{2}.$$

Let $s = \dfrac{a+b+c}{2}$, the semiperimeter of our triangle (abc);

then
$$s - a = \frac{a+b+c}{2} - \frac{2a}{2} = \frac{-a+b+c}{2},$$

$$s - b = \frac{a+b+c}{2} - \frac{2b}{2} = \frac{a-b+c}{2},$$

$$s - c = \frac{a+b+c}{2} - \frac{2c}{2} = \frac{a+b-c}{2};$$

whence by substitution,

$$c^2a^2\ \text{Cos}^2 \tfrac{1}{2} B .\ \text{Sin}^2 \tfrac{1}{2} B = s . (s-b) . (s-c) . (s-a),$$

and, extracting roots of equals,

$$ca\ \text{Cos}\ \tfrac{1}{2} B\ \text{sin}\ \tfrac{1}{2} B = \pm \sqrt{s . (s-b) . (s-c) . (s-a)}$$

$$= ca . \frac{\text{Sin } B}{2}, \text{ by [31]}; \qquad \text{C.}$$

for Sin B = '2.CHa.SH,' [31]. Now c Sin B, [H, 22], is the perpendicular on the side a of the triangle; and $\tfrac{1}{2}\,ac$ Sin B is the area, giving

$$\text{Area of } \triangle = \pm \sqrt{s . (s-a) . (s-b) . (s-c)}, \qquad \text{C.}$$

$$= \tfrac{1}{2}\,ca\ \text{Sin } B = ca . \text{Cos}\ \tfrac{1}{2} B . \text{Sin}\ \tfrac{1}{2} B.$$

In our second question,

$$s = \tfrac{1}{2}(600 + 560 + 720),$$
$$s - a = \tfrac{1}{2}(-600 + 560 + 720),$$
$$s - b = \tfrac{1}{2}(600 - 560 + 720),$$
$$s - c = \tfrac{1}{2}(600 + 560 - 720):$$

the field contains 163458·12 square feet. The double sign in the expression for the area is explained thus. Two points A and B with a third point C determine a triangle, whose area is of *one* sign when C is on one side of the line AB, of *either* sign, when C is *in* that line, and of the *contrary* sign, when C is on the contrary side of AB. To remember (C) and its demonstration, say:

RoŏP slĕb.s̆.sléc.slá, le or l for less or –; slĕb.s̆ a dissyl.
Is Síne Bang hálf cá, vid. [18].
[32] Is CHáSHca, is Area.

Write 'sqb' in quaCH and quaSHaBang,
By 'CorS is SUD'...and 'DUQ Si...'

i.e. the square *R*oot of the *P*roduct $(s-b).s.(s-c).(s-a)$ (4 syllables), = $\sin B \times \frac{1}{2}ca$ = Cos *H*alf × *S*in *H*alf (the same angle Bang) × ca = Area of the △ whose semiperimeter is s and sides a, b, c. The proof is: write 'b^2' [26 B] in terms of quaCH, the s*qua*re *C*osine of *H*alf—and again in terms of quaSHa the s*qua*re *S*ine of *Ha*lf-Bang, by aid of the formulæ [31] and [25], as in the equations a, b.

37. By retaining [32] you will be able to write down not only the area, but the Sines of the angles in terms of the sides:

(c) $$\operatorname{Sin} B \times \tfrac{1}{2} ca = \pm\sqrt{s.(s-b).(s-c).(s-a)},$$

or dividing equals by $\frac{1}{2}ca$.

$$\operatorname{Sin} B = \frac{\pm 2}{ca}\sqrt{s.(s-b).(s-c).(s-a)}.$$

By the same mnemonic you may recall the equations A and B,

$$ca.\operatorname{Cos}^2\tfrac{1}{2}B = (s-b).s, \qquad \text{A.}$$
$$ca.\operatorname{Sin}^2\tfrac{1}{2}B = (s-c).(s-a). \qquad \text{B.}$$

By multiplying together the equals on each side you get the square of (CHaSH.ca); by division of equals follows (Si by Co's tan), [22, K],

$$\tan^2\tfrac{1}{2}B = \frac{(s-c).(s-a)}{s.(s-b)}, \qquad \text{D.}$$

By going round ($bcabc$...) we deduce

$$\tan^2 \tfrac{1}{2} C = \frac{(s-a).(s-b)}{s.(s-c)}, \quad \tan^2 \tfrac{1}{2} A = \frac{(s-b).(s-c)}{s.(s-a)}. \qquad D.$$

It is sufficient to retain the last.

[33] Tásquaf A′ is slĕb. sléc by š.slá. l and le for less.

*Tan squa*re of hal*f* A = &c. Call A by its name here, not Ang. for rhyme's sake.

By going round the circle we obtain also

$$\operatorname{Sin} C = \frac{\pm 2}{ab} \sqrt{s.(s-c).(s-a).(s-b)},$$

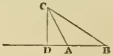
$$\operatorname{Sin} A = \frac{\pm 2}{bc} \sqrt{s.(s-a).(s-b).(s-c)}, \qquad (c)$$

$$ab . \operatorname{Cos}^2 \tfrac{1}{2} C = (s-c).s, \text{ \&c.},$$

$$ab . \operatorname{Sin}^2 \tfrac{1}{2} C = (s-a).(s-b), \text{ \&c.} \qquad A, B.$$

A step or two farther will place you in a position, so far as plane trigonometry is concerned, to attack multitudes of problems in plane geometry and practical science.

Let ABC be the angle opposite abc the sides of any triangle. Let CD be supposed perpendicular on c. If D falls within the triangle, we have by [22], CD standing for the length of the perpendicular,



$$CD = b \operatorname{Sin} A = a \operatorname{Sin} B, \text{ and}$$

if A be obtuse, so that D is *without* the triangle, we have (32) [23, L],

$$CD = b \operatorname{Sin} (\pi - A) = a \operatorname{Sin} B, \text{ i.e. by [23, L]},$$

$$b \sin A = a \operatorname{Sin} B,$$

in *either* case, and all cases,

$$\therefore \frac{b}{\operatorname{Sin} B} = \frac{a}{\operatorname{Sin} A}, \qquad e.$$

from division by $\operatorname{Sin} A . \operatorname{Sin} B$.

By placing the triangle on CB instead of AB for base, we prove the same about b, B, and c, C.

Hence $b:\operatorname{Sin} B = a:\operatorname{Sin} A = c:\operatorname{Sin} C$, E.

It follows that $\frac{b}{a} = \frac{\text{Sin } B}{\text{Sin } A}$, by multiplying by Sin $B{:}a$ in e.

wherefore $\frac{b}{a} \pm 1 = \frac{\text{Sin } B}{\text{Sin } A} \pm 1$,

or since $\pm 1 = \frac{\pm a}{a} = \frac{\pm \text{Sin } A}{\text{Sin } A}$,

$$\frac{b \pm a}{a} = \frac{\text{Sin } B \pm \text{Sin } A}{\text{Sin } A},$$

by addition of fractions:

$$\text{i.e. } \frac{b+a}{a} = \frac{\text{Sin } B + \text{Sin } A}{\text{Sin } A},$$

$$\frac{b-a}{a} = \frac{\text{Sin } B - \text{Sin } A}{\text{Sin } A}; \text{ whence by division,}$$

$$\frac{b+a}{b-a} = \frac{\text{Sin } B + \text{Sin } A}{\text{Sin } B - \text{Sin } A}.$$

Now $\text{Sin } B + \text{Sin } A = \text{Sin } \tfrac{1}{2}(B+A) . \text{Cos } \tfrac{1}{2}(B-A)$,

$\text{Sin } B - \text{Sin } A = \text{Cos } \tfrac{1}{2}(B+A) . \text{Sin } \tfrac{1}{2}(B-A)$,

by [30], 'Sa mol Sib's.'

$$\text{whence } \frac{\text{Sin } B + \text{Sin } A}{\text{Sin } B - \text{Sin } A} = \frac{\text{Sin } \frac{1}{2}(B+A) . \text{Cos } \frac{1}{2}(B-A)}{\text{Cos } \frac{1}{2}(B+A) . \text{Sin } \frac{1}{2}(B-A)}$$

$$= \frac{\text{Sin } \frac{1}{2}(B+A)}{\text{Cos } \frac{1}{2}(B+A)} \div \frac{\text{Sin } \frac{1}{2}(B-A)}{\text{Cos } \frac{1}{2}(B-A)}$$

$$= \frac{\tan \frac{1}{2}(B+A)}{\tan \frac{1}{2}(B-A)}. \qquad [22, \text{K}].$$

$$\text{i.e. } \frac{\text{Sin } B + \text{Sin } A}{\text{Sin } B - \text{Sin } A} = \frac{b+a}{b-a} = \frac{\tan \frac{1}{2}(B+A)}{\tan \frac{1}{2}(B-A)},$$

and consequently

$$\frac{\text{Sin } C + \text{Sin } B}{\text{Sin } C - \text{Sin } B} = \frac{c+b}{c-b} = \frac{\tan \frac{1}{2}(C+B)}{\tan \frac{1}{2}(C-B)}, \qquad (\text{F}).$$

$$\frac{\text{Sin } A + \text{Sin } C}{\text{Sin } A - \text{Sin } C} = \frac{a+c}{a-c} = \frac{\tan \frac{1}{2}(A+C)}{\tan \frac{1}{2}(A-C)},$$

are also true for the sides and angles of any triangle.

You may repeat (E) and (F) thus:

[34] Síde to Sinóp as síde to Sinóp (E).
[35] SŭbўD(Sínes or Sídes) is tăf Súm by tăf Díff (F).

Side to the *Sin*e of *op*posite angle, as *Side to Sin*e of *op*posite angle. *Su*m *by D*iff. of *Sines* (*or* of *Sides* opposite) = *ta*n of hal*f Su*m by *ta*n of hal*f Diff*. of angles opposite those sides.

If we divide equals by equals [30],

$$\frac{\operatorname{Sin}(\alpha \pm \tau)}{\operatorname{Cos}(\alpha \pm \tau)} = \frac{\operatorname{Sin}\alpha \,.\, \operatorname{Cos}\tau \pm \operatorname{Cos}\alpha \,.\, \operatorname{Sin}\tau}{\operatorname{Cos}\alpha \,.\, \operatorname{Cos}\tau \mp \operatorname{Sin}\alpha \,.\, \operatorname{Sin}\tau},$$

or, dividing numerator and denominator by the number $(\cos\alpha \times \cos\tau)$, which cannot change the value of the fraction on the right,

$$\frac{\operatorname{Sin}(\alpha \pm \tau)}{\operatorname{Cos}(\alpha \pm \tau)} = \frac{\frac{\operatorname{Sin}\alpha}{\operatorname{Cos}\alpha} \pm \frac{\operatorname{Sin}\tau}{\operatorname{Cos}\tau}}{1 \mp \frac{\operatorname{Sin}\alpha}{\operatorname{Cos}\alpha} \cdot \frac{\operatorname{Sin}\tau}{\operatorname{Cos}\tau}}; \qquad \text{i. e. [22, K].}$$

$$\tan(\alpha \pm \omega) = \frac{\tan\alpha \pm \tan\omega}{1 \mp \tan\alpha \,.\, \tan\omega}. \qquad \text{(G).}$$

I have put ω for τ (which makes no difference, since both are equally general), for the sake of saying more smoothly; (pron. ω as ō):

[36] tăn(ă mólŏ)'s tă mól tω bў DórS (ŭn tă. tώ).

*t*a, *t*ω, for tan α, tan ω; by DorS is by *D*iff. *or S*um, (of *un*ity and tan α . tan ω): take Diff. for $+\,\omega$, and *S*um for $-\,\omega$. This formula is true, like [30] for *any* numbers α and ω.

LESSON X.

38. *Richard*:—We have been trying to find the angles of the triangle, considered in Lesson VIII., whose sides are 6, 8, and 9, by Hutton's tables. Making decimals of the fractions, we have

$$\operatorname{Cos} B = 53{:}108 = \cdot 4907407\,;\ \operatorname{Cos} C = 19{:}96 = \cdot 1979166\,;$$
$$\operatorname{Cos} A = 109{:}144 = \cdot 7569444.$$

In the tables we find the following angles in degrees and minutes,

$\cdot 4909038 = \text{Cos}\,(60^\circ\ 36')$ and $\cdot 4906503 = \text{Cos}\,(60^\circ\ 37')$,

from which we gather that B is between $60^\circ\ 36'$ and $60^\circ\ 37'$.

$\cdot 1976573 = \text{Cos}\,(78^\circ\ 36')$ and $\cdot 1979425 = \text{Cos}\,(78^\circ\ 35')$,

so that C is between these, and nearer to the latter angle.

$\cdot 7569951 = \text{Cos}\,(40^\circ\ 48')$ and $\cdot 7568050 = \text{Cos}\,(40^\circ\ 49')$,

shewing that A is very near $40^\circ\ 48'$.

$$(60^\circ\ 37') + (78^\circ\ 35') + (40^\circ\ 48') = 180^\circ,$$

as it ought to be. We must wait for the promised lesson on the use of the tables, before we can determine the angles to a second. But I wish to know the exact *numbers* of these angles. Why is it all degrees and minutes in Hutton? You say that every arc has its own number, and every number its corresponding arc. How many *inches* are there in these arcs?

Uncle Pen.:—This you may find yourself by simple proportion. How many inches in our scale-semicircle, of 180 degrees?

Richard:—You told us *t.afalout*, i. e. π, or $3\cdot141593$; well then, if proportion will do it, it must be thus: if

$$B = 60^\circ\ 36',$$

$$180^\circ : 3\cdot1416 :: (60\tfrac{3}{5})^\circ : B \text{ in inches},$$

$$\text{or } B = \frac{60\cdot6 \times 3\cdot1416}{180} = 1\cdot057672 \text{ inches},$$

at least within the hundred thousandth part of an inch, or thereabouts.

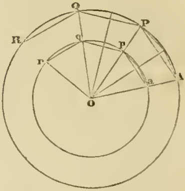
Uncle Pen.:—Generally, to reduce the *linear* measure of an arc of the scale-circle to *circular*, or circular to linear, if θ be the number of linear units, and δ that of the degrees in it, you have the proportions $\delta:\theta = 180:\pi$, and $\theta:\delta = \pi:180$, giving $\delta = 180\,\theta:\pi$, and $\theta = \pi\delta:180$. Thus if $\theta = 100$, we find δ by

$$\delta = \frac{18000}{3\cdot1416} = (5729\cdot5645)^\circ$$

$$= 15 \times 360^\circ + 329^\circ\ 33'\ 52'' = 16 \times 360^\circ - 30^\circ\ 26'\ 8''.$$

The arc whose length is 100 inches is between $15 \times 2\pi$ and $16 \times 2\pi$; it is nearly $30\frac{1}{2}$ degrees less than 16 circumferences; it has the same sine and cosine as $+(329^0\ 33'\ 52'')$ or as $-(30^0\ 26'\ 8'')$; v (32, F).

In exact reasonings about the dependence of an arc on its sine or of a sine upon its arc, it is necessary to have both expressed in linear units; and, unless the contrary is affirmed, θ or ω or ϕ, or A, B or C, signifying an arc, will always stand *in our equations* for so many inches of the circle whose radius is one inch. In practical measurements the circular division is more convenient, and for this reason is employed by Hutton. The linear measure of m degrees on *any* circle is easily found by proportion.



Let apq be any circle concentric with our scale-circle APQ, which is cut in p by the radius OP. Opa and OPA are isosceles triangles, and the bisector of the angle POA is perpendicular to both ap and AP [18], which are consequently parallel lines, cutting the legs OA, OP; hence by [6],

$$ap{:}AP = Oa{:}OA, \text{ or since } OA = 1$$

$$ap = Oa{:}AP,$$

let us suppose that AP is the side of a regular polygon of n equal sides, $APQR\ldots$, inscribed in the circle APQ, and $apqr\ldots$ the points in which the radii OP OQ $OR\ldots$ meet the concentric circle; $apqr\ldots$ will be a regular polygon of n sides also, for the angles at O will all be equal, and will stand (32) on equal arcs ap, pq, $qr\ldots$ The perimeters of the two polygons will be $n \times AP$, and $n \times ap$; and we have by the last equation $n \times ap = Oa \times n \times AP$. This must be true, however great n the number of sides of the inscribed polygon be supposed. If we suppose $n =$ a million, we shall have two polygons immeasurably near in figure and area to the two circles. We may suppose n inconceivably greater than *any* assignable number, even to infinity. When n is infinite, the polygon coincides with the circle; for a circle is a regular polygon the number of whose sides is *infinite*. It follows that, $(r = Oa)$,

A. Circumf. $apq... = Oa \times$ circumf. $APQR\ .. = 2\pi r$,

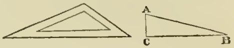
$\frac{1}{360}\cdot$ circumf. $apq... = Oa \times \frac{1}{360}\cdot$ circumf. $APQR...$,

or a degree of the circumf. $apqr = Oa \times$ a degree of the scale circle. If Oa' be the radius of any other circle, it is proved in the same way that *its* degree is Oa' times the scale-degree; or its minute, or second, &c.; i. e.

Similar arcs of two circles (i. e. *arcs on which stand equal angles at the centre*) *are proportional to the two radii.*

39. A triangle has six parts or elements, three sides and three angles. Of these if any three be given, the remaining three can be found, except the case in which the three angles are given. It is evident that any number of triangles abc, $a_1 b_1 c_1$,...may have the same three angles, if a and a_1 are parallel, as also b and b_1, c and c_1, &c.

When the triangle is right-angled at C, we require only two more data to determine the three remaining parts. The *different* cases that can occur are:



1. Given c and a,
   Required b, A, B.
2. Given a and b,
   Required c, A, B.
3. Given C and A,
   Required b, a, B.
4. Given a and A,
   Required c, b, B.
5. Given a and B,
   Required b, c, A.

1. Since $c^2 = a^2 + b^2$, [7], $c^2 - a^2 = b^2$, and $b = \pm\sqrt{c^2 - a^2}$

$\text{Sin}\, A = \frac{a}{c}$; $B = \frac{\pi}{2} - A$, (Prop. D.) or $\text{Cos}\, B = \frac{a}{c}$ [23].

From the tables, A is found by its sine, and B is its complement, for $A + B + C = \pi$ or two right angles, and $C = \frac{\pi}{2}$, or one right angle.

2. $c = \sqrt{a^2 + b^2}$; $\tan A = \frac{a}{b}$ [22] 'tan is vi (op. ad.)';

$\tan B = \frac{b}{a}$, or $B = \frac{1}{2}\pi - A$.

3. $b = c\,\text{Cos}\, A$ [22] $a = c\,\text{Sin}\, A$, $\text{Cos}\, B = a{:}c$.

4. $c = a{:}\text{Sin}\,A$, by division, from $a = c\,\text{Sin}\,A$;
$b = a\cot A$, by multiplication, from $b{:}a = \cot A$
$= 1{:}\tan A$; [24].

5. $b = a\tan B$ by mult. from $b{:}a = \tan B$
$c = a{:}\text{Cos}\,B = a\sec B$; [24], from $c\,\text{Cos}\,B = a$.

The case of c and b given, differs in nothing, more than the exchange of two letters a and b, from the first case.

When the triangle is oblique-angled, the cases are

1. Given abc, Sought ABC.
2. Given abC, Sought ABc.
3. Given ABc, Sought abC.
4. Given ABa, Sought bcC.
5. Given abA, Sought BCc.

1. Any of the formulæ (C, 34) or (cABD, 36) are sufficient. The most useful in practice are (ABD, 36), the last of which 'tasquaf A..' is used whenever the angle A does not approach 2π or 180°; and the first one when it does. *Theoretically*, all these formulæ are equally effective; i. e. supposing that no decimals are neglected in the computations.

2. Two sides and the contained angle are given. Since in every $\triangle$, by Prop. D, $A + B + C = \pi$, $A + B$ is known: and, since $\frac{1}{2}(A+B) = \frac{1}{2}\pi - \frac{1}{2}C$,

$$\text{by [22] } \tan\tfrac{1}{2}(A+B) = \frac{\text{Sin}\,(\frac{1}{2}\pi - \frac{1}{2}C)}{\text{Cos}\,(\frac{1}{2}\pi - \frac{1}{2}C)} = \frac{\text{Cos}\,\frac{1}{2}C}{\text{Sin}\,\frac{1}{2}C}, \text{ by [23, G]}$$

$$= \cot\tfrac{1}{2}C, \text{ by [24]; then}$$

$$\text{by [35] } \frac{a+b}{a-b} = \frac{\cot\frac{1}{2}C}{\tan\frac{1}{2}(A-B)}, \text{ whence}$$

$$\frac{\tan\frac{1}{2}(A-B)}{\cot\frac{1}{2}C} = \frac{a-b}{a+b};$$

for equal numbers have equal reciprocals: thus,

$$5 = \frac{10}{2}, \text{ and } \frac{1}{5} = \frac{2}{10}:$$

$$\therefore \tan\tfrac{1}{2}(A-B) = \frac{a-b}{a+b}\cdot\cot\tfrac{1}{2}C, \text{ by multiplication by } \cot\tfrac{1}{2}C;$$

this is a given number, since a and b are known, and $\cot\frac{1}{2}C$

is given in the tables. We thus have $\frac{1}{2}(A-B)$ and $\frac{1}{2}(A+B)$, and consequently both A and B by [28] '*HaS*(ab)..' (9, 36).

The side c is found by [26, B].

3. Two angles and the side between them are given. $C = \pi - A - B$ (by Prop. D); $\therefore$ $\text{Sin}\, C = \text{Sin}\,(A+B)$, and $a = c\,.\,\text{Sin}\,A{:}\text{Sin}\,(A+B)$ [34], $b = c\,.\,\text{Sin}\,B{:}\text{Sin}\,(A+B)$.

4. Two angles and a side opposite one of them is given.

$$C = \pi - (A+B),\quad b = a\frac{\text{Sin}\,B}{\text{Sin}\,A},\quad c = a\frac{\text{Sin}\,C}{\text{Sin}\,A} = a\frac{\text{Sin}\,(A+B)}{\text{Sin}\,A}.$$

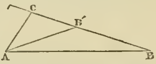
5. Two sides and an angle opposite one of them are given.

$$\frac{\text{Sin}\,B}{b} = \frac{\text{Sin}\,A}{a} \text{ gives } \text{Sin}\,B = \text{Sin}\,A\,.\,b{:}a.$$

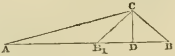
This gives the sine of B, but as B and $\pi - B$ have the same sine, it remains doubtful which of these supplementary arcs, the acute, or the obtuse, is the one sought.

We may sometimes decide it thus:

If $b < a$, $B < A$; for cut off from C on a a portion $= b$, $(CB' = CA)$: then ACB' is isosceles, and $CAB = CB'A$, which is $> CBA$; for $CB'A = CBA + CAB$, by Prop. D; i.e. *In any triangle, the greater of two sides is opposite the greater angle.* If then A is acute, the obtuse must be rejected, for no *obtuse* angle B is $< A$. If A is obtuse, the obtuse B must be rejected; for there cannot be two obtuse angles in a triangle, by Prop. D.



If $b > a$, $B > A$. Then A must be acute, and both values of B, the acute and the obtuse, may be suitable, or there may be two triangles $(AB_1C,\ ABC)$ which have the sides a, b, and the angle A.



$$B \text{ being found, } C \text{ is } \pi = A - B, \text{ and } c = \frac{\text{Sin}\,C}{\text{Sin}\,A}a,$$

or c may be expressed in terms of the data abA, thus: draw the $\perp$ CD: then $c = AD \pm DB$, according as D falls within the $\triangle$ or without it. $(BD)^2 + (DC)^2 = a^2$;

$\therefore$ $BD = \pm\sqrt{a^2 - (DC)^2} = \pm\sqrt{a^2 - (b\,\text{Sin}\,A)^2}$, or, as $AD = b\,\text{Cos}\,A$

$$c = b\,\text{Cos}\,A \pm \sqrt{a^2 - b^2\,\text{Sin}^2 A},$$

taking two like signs if D falls within the triangle, and two unlike if it falls without. The two values of c answer to the two values of B. You are already in possession of the knowledge requisite for geometry in its original etymological meaning, *the measurement of land;* for every plane figure can be divided into triangles, of which you can now discuss and solve all possible cases.

Whole provinces have often been surveyed and mapped by measuring a single line, and taking angles with a proper instrument, such as the theodolite. Let AB be a base known by exact measurement. If the angles PAB and PBA are observed, which can be easily done if P be visible, whatever

be its distance, from the points A and B, the triangle APB is known, as in case 3 of oblique triangles. Thus the lengths AP and PB and the angle between them are known by calculation, and the position of P can be fixed on the map or plan. If the point Q can be observed from any two angles of the triangle APB, Q is found in like manner with P: or if Q be visible only from P, not from A or B, and if PQ is a line known either by measurement, or as may easily happen, from calculation of some triangle of which it is one side, the angle BPQ can be observed, and QB is found as in Case 2, as well as the angles PQB and PBQ.

40. You can write the expression for the area of a triangle, if either ab and c are given, or ab and C, as in [32], for it is 'Sine Cang half ba;' or putting $\triangle$ for area,

$$\triangle = \tfrac{1}{2}\,ab \operatorname{Sin} C = \tfrac{1}{2}\,ca \operatorname{Sin} B = \tfrac{1}{2}\,bc \operatorname{Sin} A.$$

You can write it also in terms of the data of Case 3, ABc, by putting for a in $\frac{1}{2}\,ac \operatorname{Sin} B$ its value in terms of ABc, giving

$$\triangle = \tfrac{1}{2}\cdot\frac{c \operatorname{Sin} A}{\operatorname{Sin}(A+B)}\cdot c \operatorname{Sin} B = \tfrac{1}{2}\,c^2\,\frac{\operatorname{Sin} A \operatorname{Sin} B}{\operatorname{Sin}(A+B)}.$$

In fact the area can be expressed in terms of any data which are sufficient to determine the triangle.

The three perpendiculars from the angles on the sides abc are given with the sides and area, thus:

$$ap_1 = bp_2 = 2\triangle, \qquad ap_1 = cp_3 = 2\triangle,$$

wherefore $p_1 = 2\triangle{:}a$; $p_2 = 2\triangle{:}b$; $p_3 = 2\triangle{:}c$; also

$$a{:}b = p_2{:}p_1; \qquad a{:}c = p_3{:}p_1;$$

from dividing the first equation by bp_1, and the second by cp_1.

We can substitute for $\triangle$ in the values of p_1, p_2, and p_3 any expression equivalent to $\triangle$, and thus obtain various formulæ for the perpendiculars. Thus by [32],

$$p_1 = \frac{2}{a}\sqrt{s \cdot (s-a) \cdot (s-b) \cdot (s-c)}, \text{ or squaring,}$$

$$p_1^2 = \frac{4}{a^2} \cdot s \cdot (s-a) \cdot (s-b) \cdot (s-c), \text{ or}$$

$$p_1^2 = 4 \cdot \frac{a+b+c}{2a} \cdot \frac{-a+b+c}{2a} \cdot \frac{a-b+c}{2} \cdot \frac{a+b-c}{2};$$

then dividing both sides by a^2,

$$\frac{p_1^2}{a^2} = 4 \cdot \frac{a+b+c}{2a} \cdot \frac{-a+b+c}{2a} \cdot \frac{a-b+c}{2a} \cdot \frac{a+b-c}{2a};$$

for remembering that . means *times*, the denominators on the right make $2a \times 2a \times 2a \times 2a = 16a^2 \times a^2$. But the fraction $\dfrac{a+b+c}{2a}$ is not altered by dividing both numerator and denominator by a; therefore

$$\frac{p_1^2}{a^2} = 4 \cdot \frac{1+\frac{b}{a}+\frac{c}{a}}{2} \cdot \frac{-1+\frac{b}{a}+\frac{c}{a}}{2} \cdot \frac{1-\frac{b}{a}+\frac{c}{a}}{2} \cdot \frac{1+\frac{b}{a}-\frac{c}{a}}{2},$$

or multiply both sides by 4,

$$\frac{4p_1^2}{a^2} = \left(1+\frac{b}{a}+\frac{c}{a}\right) \cdot \left(-1+\frac{b}{a}+\frac{c}{a}\right) \cdot \left(1-\frac{b}{a}+\frac{c}{a}\right) \cdot \left(1+\frac{b}{a}-\frac{c}{a}\right).$$

If we now put for $\dfrac{b}{a}$ and $\dfrac{c}{a}$ the equivalent fractions just now found, $\dfrac{p_1}{p_2}$ and $\dfrac{p_1}{p_3}$,

$$\frac{4p_1^2}{a^2} = \left(1+\frac{p_1}{p_2}+\frac{p_1}{p_3}\right) \cdot \left(-1+\frac{p_1}{p_2}+\frac{p_1}{p_3}\right) \cdot \left(1-\frac{p_1}{p_2}+\frac{p_1}{p_3}\right) \cdot \left(1+\frac{p_1}{p_2}-\frac{p_1}{p_3}\right);$$

whence, extracting the square roots of these equals,

$$\frac{2p_1}{a} = \pm\sqrt{(1+p_1:p_2+p_1:p_3).(-1+p_1:p_2+p_1:p_3).(1-p_1:p_2+p_1:p_3).(1+p_1:p_2-p_1:p_3)},$$

then taking the reciprocals of these equals, and multiplying by $2p_1$,

$$a = \frac{2p_1}{\pm\sqrt{(1+p_1:p_2+p_1:p_3).(-1+p_1:p_2+p_1:p_3).(1-p_1:p_2+p_1:p_3).(1+p_1:p_2-p_1:p_3)}},$$

by which we can find the side a, if p_1, p_2, and p_3 are given.

This is the solution of the problem,

Given the three perpendiculars from the angles on the opposite sides, to find the three sides.

A more elegant exhibition of this solution is obtained by simply dividing the last equation but two by p_1^4; i. e. by $p_1 \cdot p_1 \cdot p_1 \cdot p_1$, giving after multiplication again by p_1^2

$$\frac{4}{a^2} = p_1^2\left(\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}\right)\cdot\left(-\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}\right)\cdot\left(\frac{1}{p_1}-\frac{1}{p_2}+\frac{1}{p_3}\right)\cdot\left(\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p_3}\right);$$

and thence you get $4{:}b^2$ and $4{:}c^2$, by (17) *going round* (1231...).

The squared reciprocals of $\frac{a}{2}\,\frac{b}{2}$ and $\frac{c}{2}$ being thus known, $\frac{a}{2}\,\frac{b}{2}$ and $\frac{c}{2}$ are found, and also a b and c, in terms of the three perpendiculars.

LESSON XI.

41. *Jane:*—I HAVE wished to find the triangle whose three perpendiculars are 6, 8, and 9, on the sides a, b, and c. We have by the last equation you gave us,

$$\frac{4}{a^2} = 6^2\left(\frac{1}{6}+\frac{1}{8}+\frac{1}{9}\right)\left(-\frac{1}{6}+\frac{1}{8}+\frac{1}{9}\right)\left(\frac{1}{6}-\frac{1}{8}+\frac{1}{9}\right)\left(\frac{1}{6}+\frac{1}{8}-\frac{1}{9}\right)$$

$$= \frac{6^2}{(72)^4} \times 20735.$$

$$\frac{4}{b^2} = 8^2\left(\frac{1}{8}+\frac{1}{9}+\frac{1}{6}\right)\left(-\frac{1}{8}+\frac{1}{9}+\frac{1}{6}\right)\left(\frac{1}{8}-\frac{1}{9}+\frac{1}{6}\right)\left(\frac{1}{8}+\frac{1}{9}-\frac{1}{6}\right)$$

$$= \frac{8^2}{(72)^4} \times 20735.$$

$$\frac{4}{c^2} = 9^2\left(\frac{1}{9}+\frac{1}{6}+\frac{1}{8}\right)\left(-\frac{1}{9}+\frac{1}{6}+\frac{1}{8}\right)\left(\frac{1}{9}-\frac{1}{6}+\frac{1}{8}\right)\left(\frac{1}{9}+\frac{1}{6}-\frac{1}{8}\right)$$

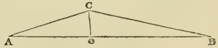
$$= \frac{9^2}{(72)^4} \times 20735.$$

20735 is $(144)^2$ nearly: so that $\frac{a}{2} = \frac{864}{144} = 6$ nearly; $\frac{b}{2} = \frac{9}{2}$ nearly, and $\frac{c}{2} = 4$ nearly; or the sides of the triangle are nearly $a = 12$, $b = 9$, $c = 8$, at least within the thousandth part of unity; a being, more exactly, 12·00029, b=9·00022, and c = 8·00019. I confess that to me these arithmetical computations are not less troublesome than the algebraic reasoning, and require as great an expenditure of time and thought.

Uncle Pen. :—It is evident that no triangle can have its perpendicular on b equal to c, except a right-angled triangle. The triangle just found is nearly right angled at A, as appears from 12^2 = nearly $9^2 + 8^2$.

Richard:—This is all very interesting about the sides and the perpendiculars. You have introduced us to *perc*, *bic*, and *biCang* [18]: can the three bisectors of the sides and angles be handled in the same manner as the perpendiculars?

Uncle Pen.:—You may readily find either the bisector of a side or that of an angle in terms of the three sides. Let $Ce = h_3$ be '*bic*,' *the bisector of c.*



By [26, B],

$a^2 = h_3^2 + (\frac{1}{2}c)^2 - h_3 c \text{ Cos } BeC$,

$b^2 = h_3^2 + (\frac{1}{2}c)^2 - h_3 c \text{ Cos } AeC$;

but AeC and BeC are supplements, therefore by (23, L)

$$a^2 + b^2 = 2h_3^2 + 2 \cdot (\tfrac{1}{2}c)^2, \text{ or dividing by } 2,$$

$$h_3^2 + (\tfrac{1}{2}c)^2 = \tfrac{1}{2}(a^2 + b^2), \text{ or}$$

$$h_3^2 = \tfrac{1}{2}(a^2 + b^2) - (\tfrac{1}{2}c)^2; \quad h_1^2 = \tfrac{1}{2}(b^2 + c^2) - (\tfrac{1}{2}a)^2;$$

$$h_2^2 = \tfrac{1}{2}(c^2 + a^2) - (\tfrac{1}{2}b)^2; \qquad \text{(B)}$$

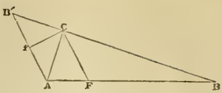
if h_1 and h_2 be the bisectors of a and b.

The square of the line drawn from any angle of a triangle to bisect the opposite side, together with the square of half that side, is equal to half the sum of the squares of the other two sides.

[37] DUQ {bic (half c)}'s half DUQ {ab}.

v. bic [18]; v. DUQ [13].

The two squares of *bic* and ($\frac{1}{2}c$) are half the two squares of a and b.



Let CF be the bisector of the angle C in the triangle ABC. Draw AB' parallel to CF, to meet CB produced in B'. The line BB', cutting the parallels CF, $B'A$, makes $FCB = CB'A = \frac{1}{2} C$, 'ext. = int. op.'; and AC cutting the same parallels makes $FCA = CAB' = \frac{1}{2} C$, 'alter. ins =' [1]. Therefore ACB' is an isosceles triangle, and Cf the perpendicular on $B'A$ bisects $B'A$ by [18], 'perc is bic:' hence

$$B'A = 2fA = 2AC \,.\, \text{Cos}\, CAF$$
$$= 2b \,.\, \text{Cos}\, \tfrac{1}{2} C.$$

Now by [6], $\dfrac{CF}{B'A} = \dfrac{BC}{BB'} = \dfrac{a}{a+b}$, or, by the preceding,

$$CF = B'A \,.\, \frac{a}{a+b} = \frac{2ba}{a+b} \text{Cos}\, \tfrac{1}{2} C; \qquad \text{C.}$$

which gives the bisector CF in terms of ab and C. By squaring equals we obtain

$$(CF)^2 = \frac{4b^2a^2}{(a+b)^2} \text{Cos}^2 \tfrac{1}{2} C: \text{ and by [32] (37, A)}$$

$ab \,\text{Cos}^2 \tfrac{1}{2} C = (s-c) \,.\, s$, whence follows by substitution,

$$(CF)^2 = \frac{4ab}{(a+b)^2} \,.\, (s-c) \,.\, s. \qquad \text{C}'.$$

This gives us the square of the bisector CF in terms of a, b and c; s being $\frac{1}{2}(a+b+c)$. If k_1, k_2, k_3 be the bisectors of A, B, and C, we thus obtain

$$k_1 = \frac{2}{b+c} \sqrt{bc \,.\, s \,.\, (s-a)}; \quad k_2 = \frac{2}{c+a} \sqrt{ca \,.\, s \,.\, (s-b)};$$

$$k_3 = \frac{2}{a+b} \sqrt{ab \,.\, s \,.\, (s-c)}. \qquad \text{C}'.$$

In the $\triangle ABC$, if $e = AF$; $BF = c - e$:

and we have [34],

$$\frac{a}{\text{Sin}\, CFB} = \frac{c-e}{\text{Sin}\, \tfrac{1}{2} C},$$

$$\frac{b}{\text{Sin}\, CFA} = \frac{e}{\text{Sin}\, \tfrac{1}{2} C},$$

whence, as Sin CFB = Sin CFA, [23, L],

$$\frac{a}{b} = \frac{c-e}{e}, \qquad \text{A.}$$

by division of equals by equals.

The following theorem is expressed in this equation (A).

The bisector of the vertical angle (C) *of any triangle* (ABC) *divides the base* (c) *into segments* {e *and* (c − e)} *which have the same ratio with that of the two sides* (b and a) *adjacent to those segments.*

42. By help of this property we can make a useful transformation of the formula for the bisector. From

$$\text{D.} \qquad \frac{a}{b} = \frac{c-e}{e}, \text{ and } \frac{b}{a} = \frac{e}{c-e}, \text{ follow}$$

$$\frac{a}{b} + 1 = \frac{c-e}{e} + 1, \text{ and } \frac{b}{a} + 1 = \frac{e}{c-e} + 1, \text{ or}$$

$$\frac{a}{b} + \frac{b}{b} = \frac{c-e}{e} + \frac{e}{e}, \text{ and } \frac{b}{a} + \frac{a}{a} = \frac{e}{c-e} + \frac{c-e}{c-e}, \text{ or}$$

$$\text{D}'. \qquad \frac{a+b}{b} = \frac{c}{e}, \text{ and } \frac{b+a}{a} = \frac{c}{c-e}.$$

Now $s.(s-c) = \frac{(a+b)+c}{2} . \frac{(a+b)-c}{2}$

$$= \frac{(a+b)^2 - c^2}{4}, \text{ [14, c]};$$

therefore $k_3{}^2 = \frac{4ab}{(a+b)^2} s.(s-c) = 4ab . \frac{(a+b)^2 - c^2}{4(a+b)^2}$,

$$\text{or } k_3{}^2 = ab . \left\{1 - \frac{c^2}{(a+b)^2}\right\} = ab - c^2 . \frac{ab}{(a+b)^2}.$$

The multiplication of the equal members of the two equations D′ gives

$$\frac{(a+b)^2}{ab} = \frac{c^2}{e(c-e)}, \text{ or } \frac{ab}{(a+b)^2} = \frac{e(c-e)}{c^2},$$

whence by multiplying equals by c^2, comes,

$$c^2 . \frac{ab}{(a+b)^2} = e(c-e), \text{ giving}$$

$$k_3{}^2 = ab - e(c-e); \text{ or } k_3 = \sqrt{ab - e(c-e)}. \qquad \text{E.}$$

DEF. *The number* 2ab:(a + b), *obtained by dividing twice the product of any two numbers* a *and* b *by their sum, is called the harmonic mean of those two numbers.*

The equations (C) and (E), with (A), may be remembered thus:

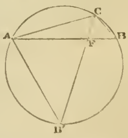
(E) BiCáng is RoóFDuP (*áb*, segs), vid. biCang [18].
[38] (C) Is CósHaCáng of HárM legs;
in CosHa pron. sh as in cash; of stands for *times*.
(A) And ví (*ab*) is the ví (segs). vid. *vi*. [5].
[39] HařM (*áb*) is twŏ *áb* by S (*áb*). S for Sum of.

As either S or M is used for SuM, so either D or F may stand for DiF, or difference. DuP is two or *duo* *P*roducts of the indicated pairs of numbers; RooF is square *Root* of Di*F*; biCang, i.e. k_3 = the square *Root* of the Di*F*ference of the *duo* (two) *P*roducts {*ab* and the *segments*, *e* and $(c - e)$, of *c*}; k_3 also is equal to the *Cos* of *Ha*lf Cang *of* the *Har*monic *M*ean of the *leg*s *a* and *b*. N.B. between two *numbers*, *of* always mean *times*, as $\frac{1}{2} \times \frac{3}{4}$, is $\frac{1}{2}$ of $\frac{3}{4}$, &c. The fourth line is the above definition.

In equations (C) and (E) are expressed the proposition following:

The bisector of any angle of a triangle, included within it, is equal to the harmonic mean of the sides containing the angle multiplied by the cosine of half the angle; and the square of that bisector is equal to the difference of the rectangles of those two sides and of the two segments of the third side made by the bisector.

If it is granted, (and it shall presently be proved) that a circle can be drawn through three given points, the property (E) can be established very simply thus:



Let the bisector CF be produced to meet at B' the circle through AB and C; join AB'; then $\angle BCF = ACB'$; $\angle CBF = CB'A$, because these are angles on the same arc AC [a, 19]. Hence the remaining pair are equal, or $\angle CFB = \angle CAB'$; for the three angles of the triangles CFB and CAB' have equal sums, and make up in each two right angles: Prop. D. We have in these triangles, which are *similar*, because they have angles of the same magnitudes, [34],

$$\frac{CF}{\text{Sin } CBF} = \frac{CB}{\text{Sin } CFB},$$

$$\frac{AC}{\text{Sin } CB'A} = \frac{CB'}{\text{Sin } B'AC},$$

whence by division,

$$\frac{\operatorname{Sin} CB'A}{\operatorname{Sin} CBF} \cdot \frac{CF}{AC} = \frac{\operatorname{Sin} B'AC}{\operatorname{Sin} CFB} \cdot \frac{CB}{CB'},$$

$$\text{or } \frac{CF}{AC} = \frac{CB}{CB'},$$

by reason of the equal angles and sines:

Hence $$CF \,.\, CB' = AC \,.\, CB,$$

by multiplying by $AC \,.\, CB'$;

$$\text{or } CF\,(CF + FB') = ba,$$

$$(CF)^2 + CF \,.\, FB' = ba,$$

$$(CF)^2 = ab - CF \,.\, FB' = ab - AF \,.\, FB = ab - e\,(c - e);$$

$$\text{for } CF \,.\, FB' = AF \,.\, FB, \text{ by [20]}.$$

In reply to Richard's enquiry about the handling of bisectors, I shall only say at present that the sides a b and c can be found by equations (B), when h_1, h_2, and h_3 are given, and from equations (C′) when k_1, k_2, and k_3 are given; but the solution of these problems, especially of the latter one, is too difficult for you to attempt, until you are more dexterous algebraists. Yet you may bear these questions in mind, as matters of future interest and enterprise.

Jane:—The device of adding unity to both sides, by which (D′) is proved, was employed in the demonstration of [35] 'SubyD Sines &c.,' and is a very pleasing contrivance.

Uncle Pen.:—It is both a simple and a fertile expedient. Let abc and $a_1b_1c_1$ be triangles having the same angles AB and C, or *similar* triangles. From $a{:}\operatorname{Sin} A = b{:}\operatorname{Sin} B$ [34], comes by dividing by $b{:}\operatorname{Sin} A$, $a{:}b = \operatorname{Sin} A{:}\operatorname{Sin} B$; and in the same way $a_1{:}b_1 = \operatorname{Sin} A{:}\operatorname{Sin} B$. We have then

$$\frac{a}{b} = \frac{a_1}{b_1}, \quad \frac{b}{c} = \frac{b_1}{c_1}, \quad \frac{c}{a} = \frac{c_1}{a_1}. \qquad \text{F.}$$

Add to each side of each ± 1, and you have as in (37),

$$\frac{a \pm b}{b} = \frac{a_1 \pm b_1}{b_1}, \quad \frac{b \pm c}{c} = \frac{b_1 \pm c_1}{c_1}, \quad \frac{c \pm a}{a} = \frac{c_1 \pm a_1}{a_1}. \qquad \text{G.}$$

Divide G with the upper sign by G with the lower, member by member;

$$\frac{a+b}{a-b}=\frac{a_1+b_1}{a_1-b_1},\ \frac{b+c}{b-c}=\frac{b_1+c_1}{b_1-c_1},\ \frac{c+a}{c-a}=\frac{c_1+a_1}{c_1-a_1}. \qquad \text{H.}$$

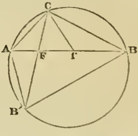
Multiplying in (F) by bb_1, &c.,

$$ab_1=a_1b,\ bc_1=b_1c,\ ca_1=c_1a; \qquad \text{F}'.$$

and then dividing by a_1b_1, &c.,

$$a{:}a_1=b{:}b_1,\ b{:}b_1=c{:}c_1,\ c{:}c_1=a{:}a_1. \qquad \text{F}''.$$

All the equations $FGHF'F''$ are true of the sides of two similar triangles.



Thus, let CF be *any* line drawn from C to meet in B' the circle through AB and C; join AB'; draw Cf making $\angle BCf=B'CA$: then the *triangles* CBf and CAB' are *similar*, like CBF and CAB' in the preceding figure. Join BB', and you have $\angle CB'B=CAF$, both on the arc CB [19], and $\angle fCA$ $(=ACB'+FCf)$, equal to the $\angle FCB=(BCf+FCf)$; wherefore the *triangles* ACf and BCB' are also *similar*.

It follows that $fB\,.\,CB'=AB'.\,CB$, and $Af\,.\,CB'=BB'.\,AC$, by the reasoning in (F) and (F′), (i.e. $ca_1=c_1a$).

Hence we have

$$CB'.\,(fB+Af)=CB\,.\,AB'+CA\,.\,BB',$$

$$\text{or}\quad CB'.\,AB=CB\,.\,AB'+CA\,.\,BB'. \qquad \text{(I)}$$

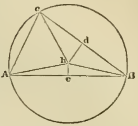
As $ACBB'$ are any four points of the circle, we have (Euclid VI. Prop. 16, Cor. 4) in the equation (I) this proposition:

The rectangle of the diagonals (AB, CB′) *of a quadrilateral in a circle, is equal to the sum of the two rectangles of the opposite sides* (AB′. CB *and* AC . BB′).

[40] In quad. inscri. two opps. are π, (c) (31).
And pró digs is DuPóppo.si. (I).

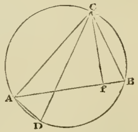
i.e. in a *quad*rilateral *inscri*bed in a circle, *two opp*osite angles are π, or together make two right angles: and the *pro*duct of the *di*a*g*onal*s* is the *du*o *P*roducts, or *do*uble *p*roduct, of the *op*posite *si*des.

43. I promised to prove that a circle can be drawn through three given points. Let them be A B and C, and let d and e be the middle points of the lines CB and AB: let the perpendiculars to those lines at d and e meet in the point h. The triangles Bhd and Chd having two sides in the one equal to two sides in the other, and the contained angles equal (34, B), have their remaining sides hB and hC equal: and in the same way, in the triangles Bhe and Ahe, the sides hA and hB are equal. The point h is therefore equidistant from the three given points: a circle having its radius $= hB$, and its centre at h, will pass through them all. The perpendicular on AC from h bisects AC by [17, a]: hence it appears that *the three perpendiculars to the sides of any triangle at their middle points meet in a point, viz. the centre of the circumscribing circle.*



To find therefore the centre of the circle which passes through three given points, it is merely necessary to join one to the remaining two, and to raise perpendiculars at the mid points of the joining lines; these perpendiculars meet in the centre required.

Let CD be the diameter of the circumscribed circle, and Cf the perpendicular from C on the side AB or c. Join AD: then CDA and CBf are similar triangles, being right angled at A and f [19], and having equal angles at D and B, and consequently the other pair of acute angles at C equal. Hence (F′, 42),



$$Cf \times CD = CB \, . \, CA, \text{ and } CD = CB \, . \, CA : Cf.$$

The same thing is proved by the consideration that the angles at D and B have equal sines. Thus we see that *the rectangle under any two sides* (CB, CA) *of a triangle is equal to that under the diameter of the circumscribing circle and the perpendicular let fall on the third side from the opposite angle.* (J).

The area is $\frac{1}{2} AB \, . \, Cf$, or $Cf = \frac{2 \triangle}{AB}$, putting $\triangle$ for area,

whence by substitution,

$$CD = CB \, . \, CA \, . \, AB : 2 \triangle, \quad \text{r } CD = abc : 2 \triangle = 2R,$$

if we put R for the radius of the circle. Hence

$$R = abc:4\triangle, \text{ or}$$

$$4R = abc:\pm\sqrt{s(s-a).(s-b).(s-c)}.$$

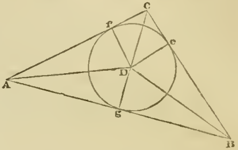
The radius of the circumscribed circle is found by dividing the product of the three sides by four times the area of the triangle. (K).

Let CD the bisector of the angle C of the triangle ABC, meet in D the bisector of A; and let Df and De be perpendiculars from D on AC and CB. By [22],

$$DC \operatorname{Sin} \tfrac{1}{2} ACB = Df = De,$$

and $DA \operatorname{Sin} \frac{1}{2} CAB = Df = Dg$,

$$\therefore\ Dg = Df = De,$$



and since the sides of the triangle are perpendicular to these lines, these sides by [17, b] are tangents of the circle which passes through gf and e, having its centre at D. Thus it is evident that a circle can be inscribed in any triangle ABC. Because $De = Dg$, if we now draw the line DB, we see that it is the bisector of the angle CBA; for [22] DBe and DBg have equal sines. Thus *the three bisectors of the angles of a triangle meet in a point, which is the centre of the inscribed circle.*

By Prop. B. (20), $AC \times Df + BC \times De + AB \times Dg =$ twice the area of the triangle ABC, i.e. putting r for De the radius,

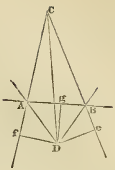
$$r(a+b+c) = 2\triangle, \text{ or, } s \text{ being } \tfrac{1}{2}(a+b+c),$$

$$r = \triangle:s. \qquad \text{(Z).}$$

The radius of the circle circumscribed about a triangle is equal to the product of the three sides divided by four times the area: and the radius of the inscribed circle is equal to the area divided by the semiperimeter.

(J) Dím. out círc is *ba* to pérc. *ba* is one syll.
[41] (Z) In. rád. of sémp. is Ar′e, Are is one syl.; of is ×.
(K) Four out. is *bac* by Are. bac a monosyl.

*Diam*eter of *out* (i.e. circumscribed) *circ*le is the quotient or ratio *ba* : *perc*. vid. [18]; *to* is here equivalent to *by*. The *in*scribed *rad*ius *of* (i.e. *times*, vid. [38 c.]) *semp*erimeter = *Are* (French for area) : *four out* (radii) = *bac by Are*a ; or $4R = bac:\triangle$.



If DA be the bisector of the angle at A below the base AB, the perpendiculars Df, Dg, and De are still equal as before: and DB being then joined is proved as before to be the bisector of the angle at B, *below the base*. Hence a circle whose centre is D and radius De will touch the sides of the triangle below the base in ef and g. We have also (Prop. B, 20),

$$AC \times Df + BB \times De - AB \times Dg = \text{twice the } \triangle\, ABC, \text{ or}$$

$r_1\,(a+b-c) = 2\,\triangle$, and exactly in the same manner,

$$r_2(a-b+c) = 2\,\triangle,$$

$$r_3(-a+b+c) = 2\,\triangle\,;$$

putting r_1 r_2 and r_3 for the radii of the cirles touching the sides below the bases c b and a. We have also, as already proved,

$$r\,(a+b+c) = 2\triangle,$$

whence by multiplication of equals,

$$\triangle^4 \text{ being } \triangle\triangle\triangle\triangle,\ 2s = a+b+c, \ \&\text{c.},$$

$$rr_1r_2r_3 \,.\, 8s\,(s-a)\,.\,(s-b)\,.\,(s-c) = 8\,\triangle^4,$$

or dividing both sides by $8\,\triangle^2$, [32],

$$rr_1r_2r_3 = \triangle^2, \text{ and } \pm\sqrt{rr_1r_2r_3} = \triangle.$$

It is usual to call the three circles whose radii are $r_1r_2r_3$ the escribed circles. You may add, if you think it will aid your memory, to the last mnemonic this line more, (*in* for *in*scribed, *e* for *e*scribed, *rad.*)

[41′] RooP. ín. e. ráds is Are; vid. RooP [32.]

which expresses the elegant theorem, that the *square root of the product of the radii of the inscribed and three escribed circles of the area of the triangle.*

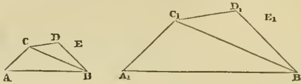
Example. If the sides are 6, 8, and 9, $\triangle = 23{\cdot}525252$, $R = 4{\cdot}59081$, $r = 2{\cdot}045673$. What are r_1, r_2, and r_3? Find them, and verify [41′].

If c be 8, we obtain from (B, 41), (D′, E, 42), $h_3 = 6{\cdot}5192024$, $e = 4{\cdot}8$, $k_3 = 6{\cdot}2161081$.

LESSON XII.

44. Def. Similar figures are such as have the same angles in the same order.

Let ABC and $A_1B_1C_1$ be similar triangles, whose sides



are abc and $a_1b_1c_1$. The areas of these triangles (40) are $\frac{1}{2}c^2 \operatorname{Sin} A \operatorname{Sin} B{:}\operatorname{Sin}(A+B)$, and $\frac{1}{2}c_1^2 \operatorname{Sin} A \operatorname{Sin} B{:}\operatorname{Sin}(A+B)$, of which numbers the quotient or ratio $= c^2{:}c_1^2$. By this, since $c^2{:}c_1^2 = a^2{:}a_1^2 = b^2{:}b_1^2$, (F″, 42) we see that *the areas of two similar triangles are proportional to the squares of their like sides,* i. e. *sides opposite the same angles.* Let now CBD and $C_1B_1D_1$ be similar triangles: $ACDB$ and $A_1C_1D_1B_1$ will be similar quadrilaterals, ACD and $A_1C_1D_1$ being equal, as sums of the same triangles, and ABD also $=A_1B_1D_1$. If we put $CB{:}C_1B_1 = N$, we have (F″, 42),

$$N = DB{:}D_1B = DC{:}D_1C_1, \text{ and also}$$

$$N = AB{:}A_1B_1 = AC{:}A_1C_1.$$

We have proved that

$$(\triangle\, CDB){:}(\triangle\, C_1D_1B_1) = N^2, \text{ i.e.}$$

$$(\triangle\, CDB) = N^2(\triangle\, C_1D_1B_1), \text{ and}$$

$$(\triangle\, ADB) = N^2(\triangle\, A_1D_1B_1);$$

whence, by addition,

$$(\triangle\, CDB + \triangle\, ADB) = N^2(\triangle\, C_1D_1B_1 + \triangle\, A_1D_1B_1), \text{ or}$$

$$\frac{(\text{fig. } ACDB)}{\text{fig. } A_1C_1D_1B_1} = N^2 = (AB)^2{:}(A_1B_1)^2 = (BD)^2{:}(B_1D_1)^2 = \&\text{c.}$$

If the figures be made similar pentagons by the addition of similar triangles DBE and $D_1B_1E_1$, it can be proved that

$$N = DB{:}D_1B_1 = BE{:}B_1E_1,$$

$$\text{and } (\triangle\, DBE) = N^2(\triangle\, D_1B_1E_1), \text{ whence}$$

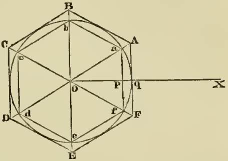
$$\frac{\text{fig.}\,(ACDB + DBE)}{\text{fig.}\,(A_1C_1D_1B_1 + D_1B_1E_1)} = N^2 = (BE)^2 : (B_1E_1)^2$$
$$= (AB)^2 : (A_1B_1)^2 = \&\text{c.}$$

In this way can be demonstrated, for any number of sides, the truth of the following proposition.

Similar polygons have their corresponding sides in a constant ratio, i. e. the ratio of every such pair is the same; and the areas of the figures are to each other as the squares of the corresponding sides.

In símils, quŏt. Ares
[42] Is vi(lik.si. squares) pron. villiksy: vid. *vi.* [5′.]

i.e. *in* two *simil*ar figures, the *quot*e of the *are*as is the quote of the *squares* of *lik*e *s*ides. Corresponding or like sides lie between like pairs of angles. This includes the former part of the proposition: for if the ratio of the *squared sides* is constant, that of the *sides* is constant.



Let $a\ b\ c\ d \ldots f$ be a regular polygon of n sides inscribed in our scale-circle; which implies that the n chords $ab, bc \ldots fa$ are all equal and that each of the n angles subtended by them at the centre is $2\pi{:}n$, or the n^{th} part of four right angles. If Op be perpendicular to af, it bisects the angle aOp [18], and the chord af, which is evidently twice the sine of aOp; or

$$ab = bc = \ldots = fa = 2\ \text{Sin}\left(\tfrac{1}{2}\,.\,\frac{2\pi}{n}\right) = 2\ \text{Sin}\,(\pi{:}n).$$

Let OA, $OB \ldots OE$, OF, be taken on Oa, Ob, &c., each equal sec $\pi{:}n$, and let AB, $BC, \ldots EF$, FA be joined, making a polygon of n sides. In the isosceles triangle OAF, Op the bisector of AOF is perpendicular on AF [18], which is therefore parallel to af; and we have by [6], $Oq{:}Op = OA{:}Oa$, or since $Oa = 1$,

$$Oq = Op\,.\,OA = \text{Cos}\,(\pi{:}n)\,.\,\text{Sec}\,(\pi{:}n) = 1\ ;\ [24]$$

wherefore q is the extremity of the radius, and the line AF perpendicular to Oq is a tangent. In the same way it can be proved that all the n sides AB, BC, $CD \ldots$ are tan-

gents parallel to the chords ab, bc, cd...; they are also all equal (B, 34); and the $\angle ABC = \angle abc$, &c.; wherefore $ABC...F$ is a regular polygon circumscribed about the circle and similar to the one inscribed.

The side AF is manifestly $= 2\tan(\pi:n)$; and the perimeter of the circumscribed polygon is $2n\tan(\pi:n)$, while that of the inscribed is $2n\sin(\pi:n)$. Wherefore

$$\frac{\text{Perimeter of inner polygon}}{\text{Perimeter of outer polygon}} = \frac{2n\,\text{Sin}(\pi:n)}{2n\tan(\pi:n)} = \cos(\pi:n).$$

When n is exceedingly great, $\pi:n$ is exceedingly small, and its cosine is immeasurably near unity, in which case the two terms of the fraction $\text{Sin}(\pi:n):\tan(\pi:n)$ differ in value by a quantity immeasurably minute, and either of these perimeters may without appreciable error be taken for the other, or for that of the circle which lies between them. It follows that if we can find correctly the sine or the tangent of an arc $\pi:n$ as small as we please, we can find the perimeter of our circle as nearly as we please, and thence by proportion that of any other circle whose radius is given. To find the value of π, the length of our scale-semicircle, is no more practicable, and no less, than to find the length of the diagonal of the square whose side is unity, i.e. the square root of 2. The value of $\sqrt{2}$ is bobodatusdipta, or 1·4142135623731, that of π is tafaloudsutuknoint, or 3·141592653589793, at least within a ten-thousand-billionth; these decimals have been extended to hundreds of places, but it is idle to seek for the termination of these infinite series of figures. There is nothing difficult, except the laboriousness of the arithmetical work, in finding the perimeter of a polygon of a million sides or more.

We know [31], $\text{Sin}\,\theta = 2\,\text{Sin}\,\tfrac{1}{2}\theta\,\text{Cos}\,\tfrac{1}{2}\theta$,

and [22], $\tan\tfrac{1}{2}\theta = \dfrac{\text{Sin}\,\tfrac{1}{2}\theta}{\text{Cos}\,\tfrac{1}{2}\theta}$;

$$\text{Sin}\,\theta\cdot\tan\tfrac{1}{2}\theta = 2\,\text{Sin}^2\tfrac{1}{2}\theta. \qquad \text{a.}$$

Also [25], $1 = \text{Cos}^2\tfrac{1}{2}\theta + \text{Sin}^2\tfrac{1}{2}\theta$,

[31], $\text{Cos}\,\theta = \text{Cos}^2\tfrac{1}{2}\theta - \text{Sin}^2\tfrac{1}{2}\theta$,

$$1 - \text{Cos}\,\theta = 2\,\text{Sin}^2\tfrac{1}{2}\theta; \qquad \text{b.}$$

Hence $\quad \operatorname{Sin}\theta \tan\frac{1}{2}\theta = 1 - \operatorname{Cos}\theta,$ c.

$$\tan\tfrac{1}{2}\theta = \frac{1 - \operatorname{Cos}\theta}{\operatorname{Sin}\theta}, \qquad c'$$

$$\tan^2\tfrac{1}{2}\theta = \frac{(1 - \operatorname{Cos}\theta)(1 - \operatorname{Cos}\theta)}{1 - \operatorname{Cos}^2\theta}$$

$$= \frac{1 - \operatorname{Cos}\theta}{1 + \operatorname{Cos}\theta}; \quad [14, c], \qquad d.$$

the numerator and denominator being both divided by $(1 - \operatorname{Cos}\theta)$.

By formula b, if we know the cosine of the angle at the centre subtended by the side of any regular polygon, we can find the sine of its half; and knowing the sine of that angle we can find the tangent of its half by equation c′. This half-angle is the angle subtended at the centre by the polygon whose sides are in number double of the first; the half of this half-angle is that at the centre under a side of the polygon having four times as many sides as the first: and thus we can proceed bisecting angles and doubling the number of sides and finding the perimeters, $2n \operatorname{Sin}(\pi{:}n)$, and $2n \tan(\pi{:}n)$, as far as we please.

45. Thus to *find the perimeter and area of the inscribed and circumscribed squares.*

Let p and P be the perimeters; we have $n = 4$, and $p = 8 \operatorname{Sin}(\pi{:}4)$ $P = 8\tan(\pi{:}4)$. In equation (b) (44), θ is the known angle at the centre, which is $\pi{:}2$, the fourth part of the circumference; and $\frac{1}{2}\theta$ is found by

$$\operatorname{Sin}^2(\pi{:}4) = \tfrac{1}{2}\left(1 - \operatorname{Cos}\frac{\pi}{2}\right),$$

$$\text{or } \operatorname{Sin}.(\pi{:}4) = \sqrt{\tfrac{1}{2}} = 1{:}\sqrt{2} = \sqrt{\cdot 5}.$$

Hence $$p = \frac{8}{\sqrt{2}} = \frac{4 . 2}{\sqrt{2}} = 4\sqrt{2}.$$

By equation c′, we have $\tan(\pi{:}4) = (1 - 0){:}1 = 1$, therefore $P = 8 \times 1 = 8$.

The area of the inscribed polygon (vid. last figure) is $n \times$ (area of triangle aOf), or [32],

$$\tfrac{1}{2}\, nOa\,.\,Of \operatorname{Sin} aOf = \tfrac{1}{2}\, n \operatorname{Sin}(2\pi:n),$$

and that of the circumscribed is

$$\tfrac{1}{2}\, nAO\,.\,OF\,.\operatorname{Sin} AOF = \tfrac{1}{2}\, n \operatorname{Sec}^2(\pi:n)\,.\operatorname{Sin}(2\pi:n).$$

$$\text{Hence } \frac{\text{Area of inner polygon}}{\text{Area of outer polygon}} = \frac{\dfrac{n}{2}\operatorname{Sin}(2\pi:n)}{\dfrac{n}{2}\operatorname{Sec}^2(\pi:n)\operatorname{Sin}(2\pi:n)}$$

$$= \frac{1}{\operatorname{Sec}^2(\pi:n)} = \operatorname{Cos}^2(\pi:n).$$

Whereby we see that these areas approach without limit to equality with each other, and consequently to the area of the circle, as n increases without limit until $\operatorname{Cos}^2(\pi:n) = \operatorname{Cos}^2 . 0 = 1$. These formulæ give us,

$$\text{since } \operatorname{Cos}\frac{\pi}{4} = \sqrt{1 - \operatorname{Sin}^2\frac{\pi}{4}} = \frac{1}{\sqrt{2}},$$

Area of inscribed square

$$= \frac{4}{2}\operatorname{Sin}\frac{\pi}{2} = 2 \text{ square units.}$$

Area of circumscribed square

$$= \frac{4}{2}\operatorname{Sec}^2\frac{\pi}{4}\operatorname{Sin}\frac{\pi}{2} = 2 : \operatorname{Cos}^2\frac{\pi}{4} = 4.$$

These are the perimeters and areas when the radius is unity. When r is the radius, we have, considering the triangle made by any side of the polygon of n sides and the radii through its extremities, in the two circles whose radii are 1 and r, if s, s' be those sides,

$$\frac{s'}{r} = \frac{s}{1}, \text{ or } s' = sr,$$

whence it follows, $ns' = r.ns$, or

perimeter for rad. $r = r \times$ perimeter for rad. unity.

In the same way the areas of the similar elementary triangles of the similar polygons in these two circles are as

$r^2 : 1^2$, [42], and area of polygon for radius $r = r^2 \times$ area for radius 1; i. e.

$$\text{area of inscribed} = \frac{nr^2}{2} \operatorname{Sin} \frac{2\pi}{n},$$

$$\text{area of circumscribed} = \frac{nr^2}{2} \operatorname{Sec}^2 \frac{\pi}{n} . \operatorname{Sin} \frac{2\pi}{n}.$$

Required the sides, perimeters, and areas of the inscribed and circumscribed regular octagons, the radius being r.

The inscribed side is $2r \operatorname{Sin}(\pi:8)$,

and $\operatorname{Sin}(\pi:8)$, by (44, b) is

$$\sqrt{\frac{1 - \operatorname{Cos}\frac{\pi}{4}}{2}} = \sqrt{\frac{1 - \sqrt{\cdot 5}}{2}}, = \cdot 3826834.$$

The circumscribed side is $2r \tan(\pi:8)$, and $\tan(\pi:8)$, by (44, c′), is

$$\frac{1 - \sqrt{\cdot 5}}{\sqrt{\cdot 5}} = \sqrt{\frac{1}{\cdot 5}} - 1 = \sqrt{2} - 1 = \cdot 4142136.$$

The perimeters are therefore

$$16r \times 0\cdot 3826834, \text{ and } 16r \times 0\cdot 4142136,$$

at least within a millionth of unity.

For the areas we have

$$\operatorname{Sin}(2\pi:8) = \sqrt{\cdot 5} = \cdot 7071068;$$

$$\operatorname{Sec}.(\pi:8) = \frac{1}{\operatorname{Cos}(\pi:8)} = \frac{1}{\sqrt{1 - \operatorname{Sin}^2(\pi:8)}}$$

$$= \frac{1}{\sqrt{1 - (\cdot 3826834)^2}} = 1\cdot 0823922.$$

The area inscribed is $4r^2 \times 0\cdot 7071068$.

The area circumscribed $= 4r^2 \times (1\cdot 0823922)^2 \times 0\cdot 7071068$.

We can now find

$\operatorname{Sin}(\pi:16)$ and $\tan(\pi:16)$, from $\operatorname{Sin}(\pi:8)$;

for $\operatorname{Cos}(\pi:8) = \sqrt{1 - \operatorname{Sin}^2(\pi:8)}$, in the formulæ (b) and (c′); and thus we determine the perimeters and areas of the outer and inner polygons of sixteen sides, and finally of

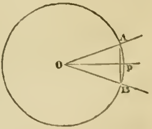
polygons that approach inconceivably near to each other and to the circle.

Jane:—I see that in Hutton's tables, the Sine and tangent are set down as equal for arcs below $18'$; thus,

$$\text{sine } 3' = \cdot 0008727 = \tan 3'.$$

Now $6'$ is $\frac{1}{3600}$ of the circumference; according to Hutton, the semi-sides and perimeters of the inscribed and circumscribed polygons of 3600 sides are sensibly alike. Then either perimeter is equal to that of the circle, at least to our perception equal. We should have 2π for the product of $2 \times 3600 \times \cdot 0008727$; but this is $2 \times 3 \cdot 14172$, and $2\pi = 2 \times 3 \cdot 141593$; how is this?

Uncle Pen.:—You have a right to say that the sine and tangent of $3'$ or $\pi{:}3600$, and therefore the sides of the polygons, are sensibly equal; for they differ certainly by less than one ten millionth of the radius, a difference which in itself is an error of little importance; but when you infer that the perimeters are also equal, you multiply that error by 3600, and it then becomes of consequence. You may however see plainly, from what we have been doing, that you can calculate π, and solve the famous problem of squaring the circle, as nearly as you please by arithmetical operations. *To square the circle,* is to find a square whose area is equal to that of the circle. The area of the inscribed polygon of n sides, when n is unlimitedly great, is the area of the circle, and this is n times the area of the elementary triangle of the polygon. This element is the product of the perpendicular into half the base,



or $Op \times Ap = \text{Cos}\,(\pi{:}n)\,.\,\text{Sin}\,(\pi{:}n)$

for radius unity,

and $= r^2\,.\,\text{Cos}\,(\pi{:}n)\,.\,\text{Sin}\,(\pi{:}n)$

for radius r, [42]. The perimeter of the polygon is $2nr\,\text{Sin}\,(\pi{:}n)$, so that the area is perimeter *times* $\frac{r}{2}\text{Cos}\,(\pi{:}n)$, which, when n is great beyond limit, making the angle $\pi{:}n$ small beyond all limit, (*that is, nothing,*) becomes = perimeter $\times \frac{r}{2}$, or (since the perimeter of the circle is $2\pi r$) $(38\,A) = \pi r^2$. That is,

C. *The area of a circle is the product of the radius and semiperimeter, and* $= \pi \times (rad.)^2$.

It is easily seen that the area of any polygon, regular or not, circumscribed about a circle, is the product of the radius and semiperimeter *of the polygon.*

To square the circle whose radius is r, we have to solve the equation $s^2 = \pi r^2$, giving $s = r\sqrt{\pi}$, the length of the side of the sought square, which is found by extraction of the root, with more or less exactness, as you take more or fewer of the decimal places in π. The important equations (b and c) may be easily remembered; thus, putting for $1 - \text{Cos}\,\theta$ its value ver. θ (Def. 5, 32);

[43]	Táf. Sin is ver.;	(c)	vid. taf. [35.]
	Two quásif is ver.	(b)	
[43]′	Rim's twó . π . rad; Are's r' . π . rad.		(A. 38.) C.

Sif means *s*ine of hal*f*, as taf is *ta*n of hal*f*: (tan of hal*f* θ) times sin θ = ver θ; and 2 (*squa*red *si*n of hal*f* θ) = ver θ. You can safely omit θ after taf, sin, sif, and ver. The *rim* or *circumf.* $= 2\pi r$; Area $= r\pi r = \pi r^2 =$ rad . π . rad; pron. *r pў rad.*

I should have remarked, just now, that you will be hereafter introduced to more expeditious methods of calculating the value of π, so that it would be a profitless labour to attempt this by the method pointed out above. There are too methods of determining the sines, cosines, &c., of arcs, somewhat more compendious and generally applicable than the one I have described; of which you may see some stated in the introduction to the tables. My object is answered for the present by giving you the information you now possess. From the known sine and cosine of 90^0 or of any other arc, the sines, cosines, &c., of all arcs can be found, by [27], [29], and [43], to any required extent of decimal places.

LESSON XIII.

46. You know that if a be any number,

$$a \cdot a = a^2 = a^{1+1}; \quad a \cdot a \cdot a = a^3 = a^{1+1+1} = a^2 a = a^{2+1},$$

$$a \cdot a \cdot a \cdot a = a^4 = a^{1+1+1+1} = a^2 \cdot a^2 = a^{2+2} = a^3 a^1 = a^{3+1};$$

and whatever whole numbers b and c may be,

$$a^b \cdot a^c = \{a \cdot a \cdot a \cdots (b \text{ factors})\}(a \cdot a \cdot a \cdots c \text{ factors}) = a^{b+c};$$

$$\text{or } (a \text{ to } b^{\text{th}}) \times (a \text{ to } c^{\text{th}}) = a \text{ to } (b+c)^{\text{th}} \text{ (power)}.$$

Further $a^4{:}a = a^3 = a^{4-1}$; $a^4{:}a^2 = a^2 = a^{4-2}$; $a^4{:}a^3 = a^1 = a^{4-3} = a$;

$$a^4{:}a^4 = a^0 = a^{4-4} = 1; \quad \text{for } a^4{:}a^4 \text{ is certainly} = 1,$$

so that we may affirm that $a^4{:}a^4 = a^0$, if by a^0 we mean unity.

$$a^4{:}a^5 = \frac{aaaa}{aaaaa} = \frac{1}{a}; \quad a^4{:}a^6 = \frac{aaaa}{aaaaaa} = \frac{1}{a^2};$$

hence we may affirm that $a^4{:}a^5 = a^{4-5} = a^{-1}$, and that $a^4{:}a^6 = a^{4-6} = a^{-2}$, if we mean by a^{-1} and a^{-2} merely the reciprocals of a and a^2. In the same way, we can always affirm

$$a^b{:}a^c = a^{b-c}, \text{ whether } b > c, \text{ or } b = c, \text{ or } b < c;$$

if we understand that $a^{b-b} = a^0$ is unity, and a^{-p} is the reciprocal of a^p. We are to remember then that a^{-p}, (a to *less* p,) = reci*p*rocal of a^p, (cip a to p).

[44] (á to lĕ p′) is cĭp (á to p′).

We have proved that a^{b+c} means $a^b \times a^c$, and a^{b-c} means $a^b{:}a^c$, for all integer values of b and c. What if they be fractions? Suppose $b = \frac{1}{2} = c$: can we maintain intelligibly that $a^{\frac{1}{2}}.a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^1 = a$? Certainly, $\sqrt{a} \times \sqrt{a} = a$; so that we may affirm $a^{\frac{1}{2}}.a^{\frac{1}{2}} = a$, if we define $a^{\frac{1}{2}}$ to mean the square root of a; and $a^{\frac{1}{3}}.a^{\frac{1}{3}}.a^{\frac{1}{3}} = a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = a$, if $a^{\frac{1}{3}}$ be $\sqrt[3]{a}$, the cube root of a. If then $b = \frac{1}{3} = c$, can we say, $a^{\frac{1}{3}}.a^{\frac{1}{3}} = a^{\frac{1}{3}+\frac{1}{3}} = a^{\frac{2}{3}}$? Unquestionably $a^{\frac{1}{3}}.a^{\frac{1}{3}} = (a^{\frac{1}{3}})^2$; for (the cube root of a) times itself must be the square of (the cube root of a). We are then safe in putting $a^{\frac{2}{3}}$ for $a^{\frac{1}{3}}.a^{\frac{1}{3}}$, if we define that

$$a^{\frac{2}{3}} = \text{the square of } a^{\frac{1}{3}} = (a^{\frac{1}{3}})^2,$$

$$\text{and it is plain that } a^{\frac{1}{r}}.a^{\frac{1}{r}} = a^{\frac{1}{r}+\frac{1}{r}} = a^{\frac{2}{r}},$$

if it be understood that

$$a^{\frac{2}{r}} = (a^{\frac{1}{r}})^2 = \text{the square of the } r^{\text{th}} \text{ root of } a.$$

Can we then say, $a^{\frac{2}{3}}.a^{\frac{2}{3}}.a^{\frac{2}{3}} = a^{\frac{2}{3}+\frac{2}{3}+\frac{2}{3}} = a^{\frac{6}{3}} = a^2$? We know well that $\sqrt[3]{a^2} \,.\, \sqrt[3]{a^2} \,.\, \sqrt[3]{a^2} = a^2$; for the same reason that

$\sqrt[3]{2}\cdot\sqrt[3]{2}\cdot\sqrt[3]{2}=2$. We can then affirm the preceding with safety, if it be laid down that

$$a^{\frac{2}{3}} = \text{the cube root of } a^2.$$

Are then the cube root of a^2, and the square of $a^{\frac{1}{3}}$ the cube root of a, the same number? To consider this, let us denote the cube root of a by y, and write the identity

A $\quad y=a^{\frac{1}{3}}$; thence, cubing equals, it follows that

$y^3=a$, and, squaring equals,

$y^3.y^3=a^2=y^6$.

The cube root of a^2 (or y^6) is clearly y^2, for $y^2.y^2.y^2=y^6$; and the square of $a^{\frac{1}{3}}$, by A, is also y^2.

i. e. the square of $a^{\frac{1}{3}}$, which is $a^{\frac{2}{3}}$, is also the cube root of a^2.

[45] ĕ tó vĭ(twŏ thrée), — v. *vi.* [5] pron. tovvy.
is Croot of sq.e, — Croot for cube root.
is squared Croot é: — sq. $e=e^2$.

i.e. $e^{\frac{2}{3}}$, *e* to vi. (2, 3), *e to* the (2:3)th power, is the *C*ube *root* of *sq*uared *e*, and is the *squared C*ube *root* of *e*. Croot e is cube *root* of *e*. You will digest this little nicety best with this mnemonic, which will prompt the meaning of $e^{\frac{m}{n}}$.

Generally, let y denote the n^{th} root of a.

From $y=a^{\frac{1}{n}}$, comes, taking n^{th} powers of equals,

$y^n=a$, and then taking m^{th} powers of equals,

$y^n.y^n.y^n\ldots(m \text{ factors})=a^m$, or $y^{n+n+\ldots(m\text{ times})}=y^{mn}=a^m$.

Now since $y^m.y^m.y^m\ldots(n\text{ factors})=y^{m+m+\ldots(n\text{ times})}=y^{nm}$, it is clear that the n^{th} root of a^m (or y^{mn}) is y^m; and the m^{th} power of $a^{\frac{1}{n}}$ is y^m, by the first equation.

Therefore the m^{th} power of $a^{\frac{1}{n}}$, the n^{th} root of a, is also the n^{th} root of a^m, the m^{th} power of a;

and $a^{\frac{m}{n}}$ is either this root or this power, whichever you please. You can easily generalize $e^{\frac{2}{3}}$ of the last mnemonic

into $e^{\frac{m}{n}}$. It is not necessary to read $e^{\frac{m}{n}}$ at length, as the m^{th} power of the n^{th} root of e, or the n^{th} root of the m^{th} power; for we may call it the $(m \text{ by } n)^{th}$ power at once, and read it e to $(m \text{ by } n)^{th}$, or e to $(m$ by $n)$. A little attention will prevent us from confounding *to* in the reading of a power with *to* in proportion. We *can* distinguish the reading of $e:p$ and e^p, as e to p and e to p^{th} (power understood). The $(m:n)^{th}$ power of a number is always given: thus $2^{\frac{2}{3}}$ the $(2:3)^{th}$ power of 2, is obtained either by squaring 2, and then taking the cube root of 4, or by finding the cube root of 2, and then squaring that root.

47. No power of a is altered in *arithmetical* value by changing merely the *form* of the index.

Then $a^{\frac{1}{2}} = a^{\frac{2}{4}} = a^{\frac{3}{6}} = a^{\frac{m}{2m}}$; for $a^{\frac{m}{2m}}$ is the m^{th} power of $a^{\frac{1}{2m}}$, i. e. of $a^{\frac{\frac{1}{2}}{m}}$, the m^{th} root of $a^{\frac{1}{2}}$; but the m^{th} power of the m^{th} root of a number is the number. You may feel, and you ought to feel satisfied, without trying to conceive exactly *how it is*, that whatever real number a^c or $a^{\frac{b}{c}}$ represents, $a^{\frac{mc}{m}}$, or $a^{\frac{mb}{mc}}$ must represent the same; and you have a right to affirm, when once the interpretation of a fractional index is fixed upon, that the $(mc)^{th}$ root of the $(mb)^{th}$ power of a is the c^{th} root of the b^{th} power of a; or that the m^{th} power of the $(b:mc)^{th}$ power of a is the m^{th} root of the $(mb:c)^{th}$ power of a. Thus we can dispense with the word root altogether, if we please: it is perfectly correct to call 12 the $(\frac{1}{2})^{th}$ power of 144, or 10 the half-power of 100. We may write $10 = 100^{0 \cdot 5}$, and $12 = 144^{0 \cdot 5}$; which assert that 10 is the 5^{th} power of the 10^{th} root of 100, and that 12 is the 10^{th} root of the 5^{th} power of 144.

We are satisfied, that for $a^b . a^c$ we can put a^{b+c}, when b and c are equal fractions; is the same allowed when they are unequal? Thus, is $a^{\frac{1}{2}} . a^{\frac{2}{3}} = a^{\frac{1}{2}+\frac{2}{3}} = a^{\frac{7}{6}}$? It is certain that $a^{\frac{1}{2}} . a^{\frac{2}{3}} = a^{\frac{3}{6}} . a^{\frac{4}{6}}$. Let $a^{\frac{1}{6}}$ the 6th root of a be denoted by y: then $y^3 . y^4$ is $a^{\frac{3}{6}} . a^{\frac{4}{6}}$, the product of the third and fourth powers of $a^{\frac{1}{6}}$: but $y^3 . y^4 = y^7$; i. e. $a^{\frac{3}{6}} . a^{\frac{4}{6}} = y^7$;

$\therefore\ a^{\frac{1}{2}} . a^{\frac{2}{3}} = y^7$ and *is* $= a^{\frac{7}{6}}$, the 7th power of y, or of $a^{\frac{1}{6}}$.

Further, $a^{\frac{1}{2}}:a^{\frac{2}{3}} = a^{\frac{1}{2}-\frac{2}{3}} = a^{-\frac{1}{6}}$; for

$$a^{\frac{1}{2}}:a^{\frac{2}{3}} = a^{\frac{3}{6}}:a^{\frac{4}{6}} = y^3:y^4 = y^{3-4} = y^{-1} = a^{-\frac{1}{6}}.$$

In the same way it can be proved that the product of *any* powers of a number, positive, or negative, whole or fractional, is the number raised to the (sum of all the indices)$^{\text{th}}$ power. Thus,

$$\frac{2^3 . 2^{\frac{1}{2}}}{2^2 . 2^{\frac{5}{6}}} = 2^3 . 2^{\frac{1}{2}} . 2^{-2} . 2^{-\frac{5}{6}} = 2^{3-2} . 2^{\frac{1}{2}-\frac{5}{6}}$$

$$= 2^{\left(3-2+\frac{1}{2}-\frac{5}{6}\right)} = 2^{1-\frac{2}{6}} = 2^{\frac{4}{6}} = 2^{\frac{2}{3}},$$

or twice the square root of 2, divided by the sixth root of 2^5 is the cube root of 4. We can thus prove,

$$x^a . x^b . x^c . x^d \ldots = x^{(a+b+c+d \ldots)},$$

where $x, a, b, c, d \ldots,$ may be any numbers of either sign.

[46] Prŏd. ány pows. of x, is
x to (sum of -dexes).

Prod. for *prod*uct; pows for *pow*ers; -dex for in*dex*. The sum is here of course the *algebraic* sum.

The product $m^3 h^3 = (mh)^3$, or the *product of the cubes* of two numbers is the *cube of the product* of the numbers, and the like is true of all powers: thus

$$mh . mh = mm . hh, \text{ or } m^2 . h^2 = (mh)^2,$$

and for any integer e,

$$(mm \ldots hh \ldots), \text{ (each } e \text{ factors)} = (mh)(mh) \ldots (e \text{ factors}),$$
$$\text{or } m^e . h^e = (mh)^e.$$

It is plain also that

$$(m^{\frac{1}{e}} h^{\frac{1}{e}})\ (m^{\frac{1}{e}} h^{\frac{1}{e}}) \ldots (e \text{ factors})$$

$$= m^{\frac{1}{e}} m^{\frac{1}{e}} \ldots (e \text{ factors}) \times h^{\frac{1}{e}} h^{\frac{1}{e}} \ldots (c \text{ factors}):$$

now $(m^{\frac{1}{e}} h^{\frac{1}{e}})$ is the e^{th} root of the product on the left, and therefore of its equal on the right, which is $m . h$;

$$\text{i.e.} \quad m^{\frac{1}{e}} h^{\frac{1}{e}} = (mh)^{\frac{1}{e}}$$

whence it appears that whether d be an integer or a fraction,

$$c^d. a^d = (ca)^d. \qquad (a)$$

This property enables us to make convenient transformations of *surds*, that is, fractional powers which cannot be exactly found.

Thus $\sqrt{12} = (4 \times 3)^{\frac{1}{2}} = 4^{\frac{1}{2}} \times 3^{\frac{1}{2}} = 2\sqrt{3}$;

$$\sqrt{12} + \sqrt{27} = \sqrt{12} + (9 \cdot 3)^{\frac{1}{2}}$$
$$= \sqrt{12} + 9^{\frac{1}{2}}. 3^{\frac{1}{2}} = (2+3)\sqrt{3} = 5\sqrt{3}.$$

As m and h above are *any* numbers, they may be

$$m = n^c, \text{ and } h = a^d,$$

so that whatever integers c d and e may be, of either sign,

$$(n^c)^{\frac{1}{e}}. (a^d)^{\frac{1}{e}} = (n^c. a^d)^{\frac{1}{e}}, \text{ or by [45], } n^{\frac{c}{e}}. a^{\frac{d}{e}} = (n^c. a^d)^{\frac{1}{e}}. \qquad (b)$$

Thus, $5^{\frac{1}{3}}.6^{\frac{1}{3}} = 30^{\frac{1}{3}}$, $5^{\frac{1}{2}}.2^{\frac{1}{3}} = 5^{\frac{3}{6}}.2^{\frac{2}{6}} = (5^3.2^2)^{\frac{1}{6}} = (500)^{\frac{1}{6}}$,

or the square root of $5 \times$ the cube root of $6 =$ the 6th root of 500.

Again,

$$5^{\frac{1}{2}}:2^{\frac{1}{3}} = 5^{\frac{1}{2}} \times 2^{-\frac{1}{3}} = 5^{\frac{3}{6}} \times 2^{-\frac{2}{6}} = (5^3 \times 2^{-2})^{\frac{1}{6}} = \left(\frac{5^3}{2^2}\right)^{\frac{1}{6}} = (31{\cdot}25)^{\frac{1}{6}},$$

or the square root of 5 divided by the cube root of 2 equal the 6th root of $31\frac{1}{4}$. From (a) follows $c^d{:}a_1{}^d = (c{:}a_1)^d$; for, if a may be any number, it may be any fraction

$$\frac{1}{a_1}; \text{ and } \left(\frac{1}{a_1}\right)^d = \frac{1}{a_1{}^d}.$$

From (b) it follows equally that

$$a^{\frac{c}{e}}{:}n^{\frac{d}{e}} = (a^c{:}n^d)^{\frac{1}{e}}.$$

It is plain that $c^{-d}. a^{-d} = (ca)^{-d}$ is the same affirmation as

$$\frac{1}{c^d} \cdot \frac{1}{a^d} = \left(\frac{1}{ca}\right)^d.$$

(a) $\begin{cases} c \text{ tód } a \text{ tód's } c\breve{a} \text{ tód,} \\ c \text{ tod by } a \text{ tod's quo. tod.} \end{cases}$ c to d, &c. quo for quote $c{:}a$.

[47] (b) $\breve{n}$ tŏ více ă tŏ víde is the e^{th} root of n' toc a tód; vice and vide are monosyll.

(c) Pór Quo. pówš is power of PorQ,
PorQuo, PorQ. is prod. or quote.

(d) Pór Quo. roóts is root of PorQ.

The first line is $c^d \cdot a^d = (ca)^d$, the second $c^d : a^d = (c:a)^d$; the third is (*b*). These three lines are examples of the general principle enunciated in the fourth and fifth lines, that the *P*roduct *or Quo*te of the *d*th *pow*ers or *roots* of two quantities *is* the *d*th *pow*er or *root of* the *P*roduct *or* Quote of the quantities.

From [47] it follows that

$$(a^d c^d)\, k^d = (ac)^d k^d = (ack)^d, \text{ and that } a^d.\, c^d : k^d = (a\,.\,c:k)^d,$$

and so on for two quantities each composed of any number of factors having the index *d*.

What are the powers of 10 between 10^1 and 10^2? Plainly they are all above 10 and below 100, and they are in number equal to that of the indices which lie between 1 and 2, i.e. they are infinite in number. Thus

$$10^{1\cdot5} \text{ or } 10^{\frac{3}{2}} = \sqrt{1000} = 31\cdot6227766;$$

therefore 31 is some power of 10 less than 1·5, and 32 is a power of the same greater than 1·5. In Hutton's tables of logarithms, you find opposite the numbers 31 and 32, the numbers 1·4913617, and 1.5051500, that is,

$$31 = 10^{1\cdot4913617},\ \ 31\cdot6227766 = 10^{1\cdot5},\ \ 32 = 10^{1\cdot50515}.$$

If then you seek the $\frac{14913617}{10000000}$th power of 10, you will obtain 31, or a number which differs from 31 by a quantity of no sensible value; i.e. 31 is the ten millionth root of 1000......000; the zeros being in number 14913617, nearly 15 millions. The number $1\cdot4913617 = \log 31$, or is the logarithm of 31 to the base 10.

Richard:—There is a sum for you, Jane; you shine in evolution. How many miles would those zeros occupy?

Jane:—I fancy that Mr Hutton did not find out by involution and evolution what power of 10 31 is; but I should greatly like to know how he discovered it. This is one of the most interesting subjects we have yet seen opened.

Uncle Pen.:—You will be introduced to this secret in due time. For the present, you may safely take it for

granted that Hutton is correct, and be satisfied with a lesson about the use of his tables, leaving the full theory of their construction till you are more advanced. And I can assure you that it is a rich remuneration for all the previous study required, to be able to understand the beautiful doctrine of logarithms.

LESSON XIV.

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48. EVERY number is some power of every other.

$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8,	are in order equal to
2^{-3}	2^{-2}	2^{-1}	2^0	2^1	2^2	2^3,	all powers of 2;
$\frac{1}{27}$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9	27,	are the foll^g. powers of 3,
3^{-3}	3^{-2}	3^{-1}	3^0	3^1	3^2	3^3;	
$\frac{1}{1000}$	$\frac{1}{100}$	$\frac{1}{10}$	1	10	100	1000,	are powers of 10, viz.
10^{-3}	10^{-2}	10^{-1}	10^0	10^1	10^2	10^3.	

The numbers between 4 and 8 are powers of 2 greater than the square and less than the cube; and so on of the rest. A list of numbers, 1, 2, 3, 4 &c., having under each the index of the equivalent power of 2, would be a table of logarithms to the base 2: and such a table can be constructed to any base. The base commonly employed is 10, and it is usual to write

$$2 = \text{com. log } 100, \quad 3 = \text{com. log } 1000, \text{ \&c.};$$

or the base is sometimes written as a subindex, thus:

$$2 = \log_{10} 100, \quad 3 = \log_3 27, \text{ \&c.},$$

read 2 is log 100 to base 10; 3 is log 27 to base 3;

but it is usual enough to omit all indication of the base when there is no risk of being misunderstood.

Whatever the base may be, 0 is always log 1; for, (46),

$$3^0 = 2^0 = a^0 = 10^0 = 1;$$

also if the base be greater than unity, all proper fractions will have a negative, and all numbers above unity a posi-

tive, logarithm. The contrary would be true if the base were a proper fraction.

The chief use of logarithms is to save the labour of multiplication and division, and to facilitate evolution and involution.

Let n and m be two numbers, suppose each of six or eight places, and let l and l_1 be their common logarithms. We have thus

$$n = 10^l,\ m = 10^{l_1},\ \text{and}\ \frac{1}{m} = 10^{-l_1} \quad [44],$$

$$nm = 10^l \,.\, 10^{l_1} = 10^{l+l_1} \text{ by } [46], \text{ and } n{:}m = 10^l \,.\, 10^{-l_1} = 10^{l-l_1};$$

$$\text{or } \log (nm) = l + l_1 = \log n + \log m,$$

$$\text{and } \log (n{:}m) = l - l_1 = \log n - \log m.$$

To find the product or quotient of n and m, we find their logarithms l and l_1 standing opposite n and m; then their product nm is seen standing opposite the logarithm $(l + l_1)$, and their quotient $n{:}m$ opposite $l - l_1$. Thus, instead of a tedious multiplication, we have only an easy addition, and, for a long division, a simple subtraction to perform. All products of two quantities (t and a), are obtained by the rule

$$\log t + \log a = \log (ta), \qquad a.$$

which includes all quotients, for

$$t{:}a = t \,.\, \frac{1}{a}, \text{ and } \log \frac{1}{a} = -\log a; \ [44],$$

so that

$$\log t + \log \frac{1}{a} = \log \left(t \,.\, \frac{1}{a}\right), \text{ or } \log t - \log a = \log (t{:}a). \qquad a'.$$

All involution and evolution is effected by this rule, for all values of p,

$$\log a^p = p \log a. \qquad b.$$

To prove this, let $p = q{:}r$, where r may be unity, if p is integer. Then if l be the logarithm of a; we have $l = \log a$,

$$\text{and } a = 10^l = 10^{\log a}$$

$$a^{\frac{1}{r}} = (10^l)^{\frac{1}{r}} = 10^{\frac{l}{r}} \text{ by } [45]$$

$$a^p = a^{\frac{q}{r}} = (10^{\frac{l}{r}})^q = 10^{\frac{ql}{r}}, \text{ or}$$

$$\log a^p = \frac{q}{r} \,.\, l = p \log a.$$

a's your báse to (fít log a)
a is ten to cóm. log a.
[48] lóg t and lóg a's lóg (ta), (a) logt a monosyl. and ta.
(lé log a for ví(ta:) (a′) le = less. vi. is quote.
lóg (a tóp) is p lŏg á. (b) { a top is a^p, a to p; top a monosyll.

Any number a is $\beta^{\log_\beta a}$, or the base β raised to the *fit* power ($\log_\beta a$), calculated for that base: $a = 10^{\text{com} \log a} = 10^{\log_{10} a}$.

49. I shall next show you how to find from the tables the logarithm of a given number, or a given arc; and, conversely, how to find the number or the arc corresponding to a given logarithm.

Let the number given be of not more than five significant places: e.g. 12345, or 1·2345, or ·0012345. Opposite 12345 in the tables of Hutton you find 0914911; this is the *decimal part* of the required logarithm, and we need no more; for we know that 12345 is between 10^4 and 10^5, that 1·2345 is between 10^0 and 10^1, and that ·0012345 is between 10^{-3} and 10^{-2}, &c.; hence we have

log 12345 = 4·0914911, log 1234·5 = 3·0914911,

log 123·45 = 2·0914911, log 12·345 = 1·0914911,

log 1·2345 = 0·0914911, log ·12345 = − 1 + ·0914911,

log ·012345 = − 2 + ·0914911, log ·0012345 = − 3 + ·0914911:

the last three are usually written thus:

$$\bar{1}\cdot0914911,\ \bar{2}\cdot0914911,\ \bar{3}\cdot0914911.$$

The integral part of the logarithm is called its characteristic or index. Use sanctions this *index of an index*.

A. The index of the logarithm of a number marks always the power of 10, positive or negative, which is *next below* the number.

Next let the given number have eight places, as 1234·5678. It is best to *consider for a moment* the first five figures as integers, and the three following as decimal places. We shall look for log 12345·678.

The tables give log 12345 = 4·0914911, and

log 12346 = 4·0915263

log 12347 = 4·0915614.

The difference of the first and second is 352, that of the second and third is 351, and log 12344 is 352 less than log 12345. Our sought logarithm is somewhere between the two first; and we must add to the first a number less than 352. As the logarithms in this part of the table manifestly increase at about the rate of 352 for every added unit in the number, we have a right to conclude that 12345·500 has its $\log = 4{\cdot}(0914911 + \frac{1}{2}352) = 4{\cdot}0915087$, or very nearly; and that the increment due to 0·678 will be given without sensible error by the proportion

$$1 : 0{\cdot}678 :: 352 : 352 \times 0{\cdot}678 = 238{\cdot}656.$$

Adding 238 to 0914911 we have

$$\log 12345{\cdot}678 = 4{\cdot}0915149,$$

and consequently

$$\log 1234{\cdot}5678 = 3{\cdot}0915149 \qquad \text{by A.}$$

Let the first five figures of our given number be called N and treated for a moment as an integer, and let the remaining figures, whether one, two, or three, be called the decimal i; we want $\log(N+i)$. The tables give $\log N$, and the difference between this and $\log(N+1)$; call this difference exhibited in the table, this *tabular difference*, $\Delta \log N$. We want now $\log(N+i) - \log N$, the quantity to be added to $\log N$ to make it $\log(N+i)$; and we obtain it by simple proportion as above, or the equation,

$$1 : i = \Delta \log N : \{\log(N+i) - \log N\},$$

which if we multiply these equals by $i \cdot \{\log(N+i) - \log N\}$ is

$$i \cdot \Delta \log N = \log(N+i) - \log N.$$

We have only to add to $\log N$ the product $i \,.\, \Delta \log N$, and $\log(N+i)$ is found, in which we have to introduce the proper index or characteristic.

If now a logarithm be given, by which we have to seek the corresponding number; we are to look for the *decimal part* of the logarithm in its place. If it is found exactly, the number is found on a line with it; if not, call the logarithm next below it $\log N$, and the given one $\log(N+i)$. We want to know i; and denoting the *difference* between $\log(N+1)$ and $\log N$, as before by $\Delta \log N$, which we obtain by subtraction, the preceding equation gives us

$$i = \{\log(N+1) - \log N\} : \Delta \log N.$$

Thus to find the number whose log is 1·2345678. The table gives

$$\log N = \log 17161 = 4{\cdot}2345426,$$

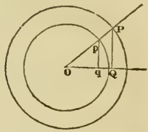
treating N for a moment as integer and the sought i as decimal. $\Delta \log N$ is 253, and $\log (N+i) - \log N$ is 252, so far as the difference of the decimals is considered only. We have then to add to N

$$i = 252:253 = {\cdot}9960\,; \quad \text{or}$$

$$4{\cdot}2345678 = \log 17161{\cdot}996, \text{ and consequently}$$

$$1{\cdot}2345678 = \log 17{\cdot}161996.$$

50. All cosines and sines, being less than the radius unity, are proper fractions, and have negative logarithms. If instead of tabulating the values of pq as $\sin \theta$, we tabulate those of PQ, the ordinate of the circle whose centre is O and radius $PQ = r$, we have always



$$PQ = OP \sin POQ = r \,.\, \mathrm{Sin}\, \theta, \qquad \text{and [48, a]}$$

$$\log PQ = \log r + \log \mathrm{Sin}\, \theta.$$

The negative value of $\log \mathrm{Sin}\, \theta$ augments numerically as $+\theta$ diminishes; for $10^{\log \mathrm{Sin}\, \theta} = \mathrm{Sin}\, \theta$; and the index becomes negative infinite when $\theta = 0$. But if r be great enough, $\log PQ$ will still be positive, for even small values of $\mathrm{Sin}\, \theta$. To avoid negative indices of logarithms, Hutton takes $OP = 10^{10}$, = ten thousand millions, instead of $Op = 10^0 = 1$, for the radius of the circle; and thus the smallest arc he has to consider, $\theta = 1''$, has an ordinate $PQ > 1$, and therefore $\log PQ > 0$. He tabulates this logarithm, which is 10 too great, for

$$\log \mathrm{Sin}\, \theta = \log PQ - \log 10^{10} = \log PQ - 10.$$

And every circular logarithm, $\log \sin \theta$, $\log \cos \theta$, &c. is too great by $\log 10^{10}$ or 10, in his tables. We have then always to subtract this 10, which is added for typographical convenience and correctness, from the tabulated value, in order to obtain the accurate logarithm of any circular function of θ. An example or two will suffice for illustration.

To find $\log\cos 6^\circ.\ 8'.\ 42''$. Call this $\log\cos(N+i)'$, i' being $0{\cdot}7' = 42''$.

We read

$$\log\cos 6^\circ.\ 8' = 9{\cdot}9975069 = \log\cos N', \text{ and}$$

$$\log\cos 6^\circ.\ 9' = 9{\cdot}9974933 = \log\cos(N+1)', \text{ whence}$$

$$\log\cos 6^\circ.\ 9' - \log\cos 6^\circ.\ 8' = -136$$

$$= \Delta\log\cos N' = \log\cos(N+1)' - \log\cos N'.$$

As the $\log\cos\theta$ decreases at about this rate, -136 for each added minute, in this part of the table, we obtain the decrements due to $0{\cdot}7'$ by proportion, as before.

We want to know

$$\log\mathrm{Cos}(N+i)' - \log\mathrm{Cos}\, N';$$

and we have it by

$$\log\mathrm{Cos}(N+i)' - \log\mathrm{Cos}\, N' : -136 :: 0{\cdot}7' : 1', \text{ or}$$

$$\log\mathrm{Cos}(N+i)' - \log\mathrm{Cos}\, N' = (\Delta\log\mathrm{Cos}\, N)\times i$$

$$= -136\times 0{\cdot}7 = -95{\cdot}2.$$

Adding this decrement -95 to $\log\mathrm{Cos}\, N'$, we obtain, by deducting 10,

$$\log\mathrm{Cos}(N+i)' = \log\mathrm{Cos}\, 6^\circ.\ 8'.\ 42''$$

$$= 9{\cdot}9974974 - 10 = -\bar{1}{\cdot}9974974.$$

You see that this is exactly the process by which we find the logarithm of any given number. The odd seconds are our i, and are to be expressed as a fraction or decimal of a minute.

To find the arc by the logarithm of its sine or cosine, &c.; as to find that whose log cosine $= 9{\cdot}9974974$, or more correctly, whose log Cos is $\bar{1}{\cdot}9974974$. We want i in the arc $(N+i)'$; knowing N from

$$\log\mathrm{Cos}\, N' = \log\mathrm{Cos}\, 6^\circ.\ 8' = 9{\cdot}9975069, \text{ by table;}$$

$$\log\mathrm{Cos}(N+i)' = 9{\cdot}9974974, \text{ as given.}$$

$$\log\mathrm{Cos}(N+i)' - \log\mathrm{Cos}\, N' = -95;$$

$$\text{also } \Delta\log\mathrm{Cos}\, N' = -136, \text{ by the table,}$$

therefore by the equation,

$$log\ Cos\ (N+i)' - log\ Cos\ N' = (\Delta\ log\ Cos\ N)\ i',$$

$$i' = \frac{-95}{-136} = 0{\cdot}698' = 42'' \text{ nearly, whence}$$

$$6^0.\ 8'.\ 42'' \text{ is the arc required.}$$

If you digest this well in the shape following, you will have a sufficient general knowledge of the use of the tables. For more detailed information, particularly about the method of finding i when the tabular difference $\Delta \log N_1$, varies greatly from minute to minute, you can consult the introduction to the tables.

	You want to fi',	fi for find.
	Dilógs or *i*.	Di = Diff.
	and Di {lŏS(Ni) lóN} is (táb. diff.) *í*:	
[49]	first take *i* for point *i*.	vid. *S*. [14], lo = log.
	Nég. is ló. of frac. pro.	
	Pow'.ten néxt sub num. is -dex.	(A, 49)
		sub = below.

You always *want to fi*nd, after the first inspection, either the *Di*f of *logs* $\log(N+i) - \log N$, when $N+i$ is given, *or* the decimal *i*, when $\log(N+i)$ is given: *and* both these are found equally from the equation

$$\log(N+i) - \log N = (tab\text{ular } diff.) \times i: \qquad \text{(line third.)}$$

loS(Ni) = *log S*um $(N+i)$. Di(loS loN) = the *Diff.* (*log S*um − *log N*). You may first take *i* for 0·*i*, point *i*, as in our first example. The numbers 0·1, 5·9, &c., are best read, point 1, 5 point 9, &c. *Neg*ative is the *log* of a *fract*ion *pro*per. The *pow*er of *ten next sub* (= below) *the num*ber is marked by the in*dex* of its log.

It should have been remarked, that, as there are no negative logarithms in the table, in order to find the number answering to such, as to $-3{\cdot}5678921$, it must be written under the equivalent form $\bar{4}{\cdot}4321079$, for, (2),

$$-3{\cdot}5678921 = {\cdot}4321079 - 4{\cdot}0000000.$$

Note also that the numbers whose logs. are in the table are *positive* numbers: of the logs. of negative numbers you have some curious knowledge to acquire hereafter.

LESSON XV.

51. You can now comprehend and remember the solutions of innumerable questions of the greatest practical importance and scientific interest.

Ex. 1. A triangular field has its sides

$$a = 1050, \quad b = 600, \quad \text{and } c = 500 \text{ yards:}$$

required the side of an equilateral triangle of equal area.

The angle of an equilateral triangle is $\frac{1}{3} \times 180^0 = 60^0$; and if z be the sought side we must have by [32],

$$\tfrac{1}{2} z^2 \operatorname{Sin} 60^0 = \sqrt{s.(s-a).(s-b).(s-c)},$$

or, *dividing both by* $\frac{1}{2}$ *Sin* 60,

$$z^2 = \frac{2}{\operatorname{Sin} 60}\{s.(s-a).(s-b).(s-c)\}^{\frac{1}{2}},$$

and, *extracting square roots of equals*,

$$z = (2{:}\operatorname{Sin} 60^0)^{\frac{1}{2}}\{s.(s-a).(s-b).(s-c)\}^{\frac{1}{4}}.$$

$$(e^{\frac{1}{4}} = e^{\frac{1:2}{2}}; \text{ [45] generalized, or } e^{\frac{1}{4}} = \sqrt{\sqrt{e}}.)$$

If a perpendicular is let fall from an angle on the opposite side of the equilateral, it bisects that side, by [18]; and [22],

$$z \operatorname{Cos} 60^0 = \tfrac{1}{2} z, \text{ whence } \operatorname{Cos} 60^0 = \tfrac{1}{2},$$

$$\operatorname{Cos}^2 60 + \operatorname{Sin}^2 60 = 1, \text{ [25]},$$

$$\text{and } \operatorname{Sin} 60 = \sqrt{1 - \tfrac{1}{4}} = \tfrac{1}{2}\sqrt{3}; \text{ wherefore}$$

$$z = (4{:}\sqrt{3})^{\frac{1}{2}}\{s.(s-a).(s-b).(s-c)\}^{\frac{1}{4}}$$

$$= \frac{2}{3^{\frac{1}{4}}} . s^{\frac{1}{4}}(s-a)^{\frac{1}{4}}(s-b)^{\frac{1}{4}}(s-c)^{\frac{1}{4}}, \text{ by [47, d], and [48]},$$

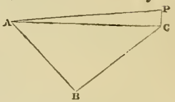
$$\log z = \log 2 - \tfrac{1}{4}\log 3 + \tfrac{1}{4}\log s$$
$$+ \tfrac{1}{4}\log(s-a) + \tfrac{1}{4}\log(s-b) + \tfrac{1}{4}\log(s-c).$$

$$s = \tfrac{1}{2}(a+b+c) = 1075, \quad s-a = 25, \quad (s-b) = 475, \quad s-c = 575.$$

$$\log 2 = 0{\cdot}3010300$$
$$\tfrac{1}{4}\log 1075 = 0{\cdot}7578521$$
$$\tfrac{1}{4}\log 25 = 0{\cdot}3494850$$
$$\tfrac{1}{4}\log 475 = 0{\cdot}6691734$$
$$\tfrac{1}{4}\log 575 = 0{\cdot}6899169$$
$$2{\cdot}7674574$$
$$-\tfrac{1}{4}\log 3 = -\,0{\cdot}1192803$$
$$\log z = \quad 2{\cdot}6481771 = \log 444{\cdot}8126,$$
$$\text{and}\quad z = 444{\cdot}8126 \text{ yards},$$

the side of the equilateral required.

Ex. 2. CP is a perpendicular cliff, surmounted by a beacon P, seen across a creek from A. Wishing to know the height PC, and the distance AC, I measure $AB = 500$ yards level with C, and I take the angles



$$PAC = 5^\circ.\,7'.\,30'',\ CAB = 45^\circ.\,48'.\,7'',$$

and $CBA = 94^\circ.\,2'.\,9''$. Required PC and AC.

The angle

$ACB = 180^\circ - (139^\circ.\,50'.\,16'') = 40^\circ.\,9'.\,44''$, by Prop. D.

By [34], $AC = AB\,.\,\text{Sin } 94^\circ.\,2'.\,9'' : \text{Sin } 40^\circ.\,9'.\,44''$,

$$\log AC = \log AB + \log \text{Sin } 94^\circ.\,2'.\,9'' - \log \text{Sin } 40^\circ.\,9'.\,44'',$$

or, since $\text{Sin}\left(\dfrac{\pi}{2} + \theta\right) = \text{Cos}\,(-\theta) = \text{Cos}\,\theta$, [23, G],

$$\log AC = \log AB + \log \text{Cos } 4^\circ.\,2'.\,9'' - \log \text{Sin } 40^\circ.\,9'.\,44'',$$

$$\log AC = 2{\cdot}6989700 + \bar{1}{\cdot}9989216 - \bar{1}{\cdot}8095286 = 2{\cdot}8883630.$$

$\therefore\ AC = 773{\cdot}3267$ yards, by the table of logs.

Next $CP = AC \tan 5^\circ.\,7'.\,30''$,

$$\log CP = \log AC + \log \tan 5^\circ.\,7'.\,30''$$
$$= 2{\cdot}8883630 + \bar{2}{\cdot}9527310 = 1{\cdot}8410940$$
$$= \log 69{\cdot}3576,$$

$\therefore\ CP = 69{\cdot}3576$ yards.

Ex. 3. A and B are two light-houses, neither visible from the other. AC and CB are two level roads making the angle $ACB = 61^\circ. 32'$, $AC = 4$ miles, $BC = 8{\cdot}75$ miles. The distance AB is required to be found by logarithms.

$$c^2 = a^2 + b^2 - 2ab \operatorname{Cos} C, \text{ [26, B]},$$

gives the value of c or AB; but this is not adapted for logarithms, which are of use only in finding products and quotients. You can obtain a^2b^2, but not $a^2 + b^2$, by the logarithms of a and b. A convenient formula is obtained from the preceding, thus, by [31] and [25],

$$c^2 = a^2 + b^2 - 2ab\,(\operatorname{Cos}^2 \tfrac{1}{2} C - \operatorname{Sin}^2 \tfrac{1}{2} C)$$

$$= a^2 + b^2 - 2ab\,(2 \operatorname{Cos}^2 \tfrac{1}{2} C - 1),$$

$$\text{or } c^2 = a^2 + b^2 + 2ab - 4ab \operatorname{Cos}^2 \tfrac{1}{2} C,$$

$$= (a+b)^2 - 4ab \operatorname{Cos}^2 \tfrac{1}{2} C, \text{ by [14]},$$

$$= (a+b)^2 \left\{1 - \frac{4ab}{(a+b)^2} \operatorname{Cos}^2 \tfrac{1}{2} C\right\},$$

for the same reason that $8 - 3 = 8\,\{1 - \frac{3}{8}\}$.

Now I say that $4ab$ is not greater than $(a+b)^2$, for let it be supposed that it is greater by c, c being positive:

$$\text{then } a^2 + 2ab + b^2 = 4ab - c, \text{ or, transposing } 4ab,$$

$$a^2 - 2ab + b^2 = -c, \text{ or } (a-b)^2 = -c, \text{ [14]},$$

which is absurd; for the negative quantity $-c$ cannot be the square either of $(a - b)$ or of $(b - a)$, (v. 23).

Therefore $4ab:(a+b)^2$ is not an improper fraction, and consequently $\operatorname{Cos}^2 \frac{1}{2} C \,.\, 4ab:(a+b)^2$ is a proper and still smaller fraction, the value of which is known. We may call it $\operatorname{Sin}^2\gamma$, and we can find by the tables the arc γ whose $(\text{sine})^2$ is this fraction. We have now the equation

$$c^2 = (a+b)^2 \{1 - \operatorname{Sin}^2\gamma\} = (a+b)^2 \operatorname{Cos}^2\gamma, \text{ [25]};$$

$$\therefore\ c = (a+b) \operatorname{Cos} \gamma, \tag{a}$$

and $\log c = \log (a+b) + \log \operatorname{Cos} \gamma - \log 10^{10}$.

In our problem

$$ab = 35,\ a + b = 12{\cdot}75;\ \tfrac{1}{2} C = 30^\circ. 46',$$

$$\operatorname{Cos}\tfrac{1}{2}C.2\sqrt{ab}:(a+b) = \operatorname{Cos} 30^{\circ}.\,46'$$
$$\times 2\sqrt{35}:12{\cdot}75 = \operatorname{Sin}\gamma, \qquad \text{(b)}$$

$$\log \operatorname{Sin}\gamma = \log \operatorname{Cos} 30^{\circ}.\,46' + \log 2 + \tfrac{1}{2}\log 35 - \log 12{\cdot}75$$
$$= 9{\cdot}9341234$$
$$+ \ {\cdot}3010300$$
$$+ \ {\cdot}7720340$$

$$11{\cdot}0071874 - 1{\cdot}1055102 = 9{\cdot}9016772$$
$$= \log \operatorname{Sin} 52^{\circ}.\,52'.\,58'', \ \therefore\ \gamma = 52^{\circ}.\,52'.\,58'',$$
$$\log \operatorname{Cos}\gamma - 10 = \bar{1}{\cdot}7806407$$
$$\log 12{\cdot}75 = 1{\cdot}1055102$$
$${\cdot}8861509 = \log 7{\cdot}69398\,;$$
$$\therefore\ c = AB = 7{\cdot}69398 \text{ miles.}$$

This is a much readier mode of finding c, especially when a and b are large numbers, than the formula [26, B], and it must be remembered. Putting Si.γ and Co.γ for Sin γ and Cos γ, say: (pron. γ like g)

[50] The Síde c' is Cóγ Sum ($\acute{a}b$), (a)
Where Síγ meăn ($\acute{a}b$)'s CHăCáng. HarM ($\acute{a}b$). (b)

vid. mean (ab) [8]; HarM (ab) v. [39]; CHa, v. [31], Cang [18].

To determine the side c in terms of a, b, and C by logarithms, we use the equation above (a), in which γ is given by

$$\operatorname{Sin}\gamma\,.\sqrt{ab} = \cos\tfrac{1}{2}C.2ab:(a+b), \qquad \text{the equation } (b) \text{ above,}$$

with the multiplier $\sqrt{ab}$ introduced on both sides, for mnemonical convenience. Vid. (15).

If we had wished to obtain the angles CAB and CBA in terms of abC, and not the side c, we have them by [35] and case 2, (39):

$$\log\tan\tfrac{1}{2}(A-B) = \log(a-b) - \log(a+b) + \log\cot\tfrac{1}{2}C$$
$$= \log 4{\cdot}75 - \log 12{\cdot}75 + \log\cot 30^{\circ}.\,46'$$
$$= 0{\cdot}6766936 - 1{\cdot}1055102 + 10{\cdot}2252412$$
$$= 9{\cdot}7964246 = \log\tan 32^{\circ}.\,2'.\,15'';$$

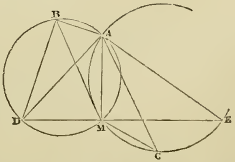
$$\therefore\ \tfrac{1}{2}(A-B)=32^\circ.\,2'.\,15'',$$
$$\tfrac{1}{2}(A+B)=59^\circ.\,14'.\,0'';$$
$$\text{for } \tfrac{1}{2}(A+B+C)=90^\circ, \text{ (Prop. D).}$$

Therefore $A=91^\circ.\,16'.\,15''$ and $B=27^\circ.\,11'.\,45''$, [28].

Richard:—The right side of (b) as written last, is 'CosHaCang of HarM.legs,' or biCang [38]; so that I could make a shorter mnemonic still, by saying *Where Siγ mean(ab)'s biCang,* instead of the second line of [50].

Uncle Pen.:—You are right: and the readiness with which you have observed this, proves both the power of these mnemonics, as instruments of rapid thought, and the attention you have paid to them. Here then you might make a little theorem, if it were worth while: *the bisector of any angle of a triangle is never less than the mean proportional of the containing sides.* This is evident from the consideration that no sine can be greater than unity. If $\operatorname{Sin}\gamma=1$, $\operatorname{Cos}\gamma=0$, and the base $c=0$, by (a); this is the case of $C=0$, and gives no triangle at all. Our theorem then is: *the bisector of an angle of a triangle is always less than the mean proportional of the sides.* Scores of such theorems can easily be made. But it is a great and common mistake, to lose about triangles and circles that time and labour, which would be more profitably employed in mastering the magical secrets of the more advanced analysis.

Ex. 4. B, A and C are three lights ashore, whose mutual position is known; M is a sand-bank, whose exact position is to be determined by observations thereat made of the two angles, $BMA=\theta$, and $AMC=\phi$.



Let $c=BA$, $b=AC$, A_1 = angle BAC, all numbers already supposed known. Let AD be the unknown diameter through A of the circle passing through B, A, and M, and AE the unknown diameter of that through A, C and M. Because by [19], AMD and EMA are both right angles, DME is a straight line to which AM is perpendicular.

$$\angle BDA = \angle BMA = \theta \quad \text{by [19]},$$
$$CAE = \quad CME = \phi - \tfrac{1}{2}\pi$$
$$= AMC - AME$$
$$AD = AB{:}\text{Sin } AMB = c{:}\text{Sin } \theta, \quad [22],$$
$$AE = AC{:}\text{Sin } AMC = AC{:}\text{Sin } AEC, [40] \text{ and } [23,\text{L}],$$
$$= b{:}\text{Sin } \phi,$$

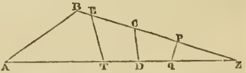
$$\therefore \log AD = \log c - \log \text{Sin } \theta\,;\ \log AE = \log b - \log \text{Sin } \phi,$$

by which AD and AE are readily found. In actual computation it will of course be necessary to deduct 10 from the tabular log Sin θ, and log Sin ϕ. Knowing now in the triangle DAE the sides AD and AE, and the contained angle, which is $A_1 + \phi + \theta$; for

$$DAE = BAC + CAE - DAB$$
$$= A_1 + \phi - \tfrac{1}{2}\pi - (\tfrac{1}{2}\pi - \theta) = A_1 + \phi + \theta\,;$$

we can find by case 2 of oblique triangles (39) the base angle ADE, which gives us $DAM = \frac{1}{2}\pi - ADE$, and finally $AM = AD$ Sin ADE. Also $BAM = BAD + DAM$ is known, being $= BAC + CAE - DAE + DAM$, the first of which four is previously given, while the remaining three have been found. Knowing thus AM in length, and the angle BAM, we have determined the point M, which can be laid down on a chart. Vid. Legendre, *Traité de Trigonométrie*, § LXI.

Ex. 5. A quadrilateral field $ABCD$ is to be divided in a given proportion by a line drawn from T a point in AD one of its sides.



Let the sides BC and AD be produced to meet in Z; the area of the triangle CDZ can be found by [32], and also that of the field: let these areas be called $\triangle$ and F, and let f be put for the distance TZ, which can be found by measurement. We need to know the angle CZT, which we shall call θ. If the angles of the field are known, θ is the supplement of the $\angle (A + B)$ the sum of those at A and B (Prop. D), and is given; or we can observe θ by a proper instrument for taking angles; or we may raise at any point p in BZ a perpendicular pq by *the bricklayers' theorem* [7], and then, measuring pq intercepted between

the produced sides, and also qZ, we have $\operatorname{Sin}\theta = pq:qZ$. The portion $ABET$ of the field to be cut off is m times F, m being a known number less than unity. Call the sought distance ZE, x: we can express that the triangle EZT is to be $= \triangle + ECDT$, thus, $ECDT$ being $F - mF$,

$$\tfrac{1}{2}\,xf.\operatorname{Sin}\theta = \triangle + F - mF = \triangle + (1-m)F, \text{ whence comes}$$

$$x = \frac{2}{f\operatorname{Sin}\theta}.\{\triangle + (1-m)F\}, \text{ by division of equals,}$$

which gives ZE in known numbers, and TE can be drawn.

To find the area of a quadrilateral, it is merely necessary to draw a diagonal, making two triangles whose areas can be determined by [32]. If both diagonals are known, h and k, inclined at the angle ϕ, no matter whether obtuse or acute, the whole figure is the sum of the four triangles having their common vertex at s, the intersection of the diagonals, and is

$$= \tfrac{1}{2}\operatorname{Sin}\phi\,(As\,.\,Ds + As\,.\,Bs + Cs\,.\,Ds + Cs\,.\,Bs) = \tfrac{1}{2}\operatorname{Sin}\phi\,.\,hk.$$

Thus [32] *the area of a quadrilateral is equal to that of a triangle whose sides are the two diagonals inclined at the angle between the diagonals, either the obtuse, or the acute angle.* It is plain, from the consideration that $\operatorname{Sin}\phi = \operatorname{Sin}(\pi - \phi)$, that *two* distinct triangles can be made to have two given sides and to contain a given area, except when that area is half the product of the two sides; in which case $\phi = \pi - \phi = \frac{1}{2}\pi$.

Ex. 6. The base (c) of a triangle, the difference (δ) of the two angles at the base, and the difference (d) of the sides being given; find the triangle.

Let $c =$ the base, $\delta = B - A$, the given difference of angles, and $d = b - a$, the given difference of the sides.

We shall know B and A by [28] if we can find $\sigma = B + A$; for $B = \frac{1}{2}(\delta + \sigma)$, and $A = \frac{1}{2}(\sigma - \delta)$.

Hence by [34] [23], $b = \dfrac{c\operatorname{Sin}\frac{1}{2}(\sigma + \delta)}{\operatorname{Sin}\sigma}$, for $C = \pi - \sigma$;

$$a = \frac{c\operatorname{Sin}\frac{1}{2}(\sigma - \delta)}{\operatorname{Sin}\sigma};$$

$$\therefore\ b - a\,(=d) = \frac{c}{\text{Sin}\,\sigma}\cdot\left\{\text{Sin}\left(\frac{\sigma}{2}+\frac{\delta}{2}\right) - \text{Sin}\left(\frac{\sigma}{2}-\frac{\delta}{2}\right)\right\}$$

$$= \frac{c}{\text{Sin}\,\sigma}\cdot 2\,\text{Cos}\,\frac{\sigma}{2}\,.\,\text{Sin}\,\frac{\delta}{2}\quad \text{by [29]}$$

$$= c\,.\,\frac{2\,\text{Cos}\,\frac{\sigma}{2}\,\text{Sin}\,\frac{\delta}{2}}{2\,\text{Sin}\,\frac{\sigma}{2}\,\text{Cos}\,\frac{\sigma}{2}}\quad \text{by [31]},$$

$$\text{or}\ \ d = \frac{c\,.\,\text{Sin}\,\frac{\delta}{2}}{\text{Sin}\,\frac{\sigma}{2}},$$

whence, multiplying by $\text{Sin}\,\frac{\sigma}{2} : d$,

$$\text{Sin}\,\frac{\sigma}{2} = \frac{c\,.\,\text{Sin}\,\frac{\delta}{2}}{d};$$

$$\log\,\text{Sin}\,\frac{\sigma}{2} = \log c + \log\,\text{Sin}\,\frac{\delta}{2} - \log d.$$

Thus $\frac{\sigma}{2}$ is found and thence $A = \frac{\sigma}{2} - \frac{\delta}{2}$, and $B = \frac{\sigma}{2} + \frac{\delta}{2}$.

Vid. Hirsch's *Geometry*, § LXVIII.

LESSON XVI.

52. *Jane:*—We have heard nothing of the co-ordinates of Des Cartes, for many lessons. I should like to know how far these questions about distances and areas can be solved, when the data are all lines measured parallel to axes of x and y, and how the answers would be both expressed in terms of given points $(x_1 y_1)$, $(x_2 y_2)$, &c.

Richard:—If the points are given 'in co-ords recta,' we know by [12] all the joining lines, and we can put for every symbol of a given length the equivalent *poth.*,

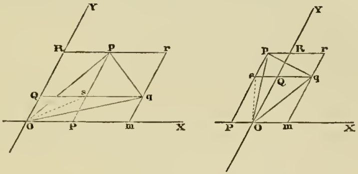
$$\pm\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

All our results would then have the shape you wish to see, would they not?

Jane:—So far as distances enter into them, they would, and a very inconvenient shape it would be, with so many square roots in it.

Richard:—The sines and cosines of angles also might be represented in the same manner, for these are expressed in [32] and [33] in terms of the sides a, b, c, about them. How do you express the area of a triangle in terms of the co-ordinates of the three angles?

Uncle Pen.:—This is worth knowing. Let the origin



be one of the three angles, and q, $(x_1 y_1)$ and p, $(x_2 y_2)$, be the other two. Let x_2 and y_1 through q and p meet in r, and x_1 meet y_2 in s: join Os. The triangle $Oqp = pqs + Oqs + Ops$, if all the co-ordinates are positive, as in the first figure, and $= pqs + Ops - Oqs$, if x_2 is negative, as in the second. The triangle pqs is half the parallelogram $sprq$ (D 34); the triangles Ops and Oqs are halves of the parallelograms $RpsQ$ and $sqmP$, being on the same bases ps, qs, with them, and between the same parallels, i.e. having the same altitudes (Prop. B, 20). Therefore

$$\triangle\, Oqp = \tfrac{1}{2}\{prqs + sqmP + RpsQ\} \text{ in one figure, and}$$

$$= \tfrac{1}{2}\{prqs + sqmP - RpsQ\} \text{ in the other:}$$

the first is $\frac{1}{2}(ORrm - OQsP)$; the second is $\frac{1}{2}(prmP - RpsQ)$, or $\frac{1}{2}(ORrm + OQsP)$.

From (D 34) and [32], it follows *that the area of a parallelogram whose sides* a *and* b *include the angle* ω *is* ab . Sin ω. (E.)

If then $YOX = \omega$, we see that as

$$Om = x_1, \quad OR = y_2, \quad OQ = y_1, \quad PO = x_2,$$

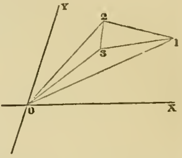
$$\triangle\, Oqp = \tfrac{1}{2} x_1 y_2 \operatorname{Sin}\omega - \tfrac{1}{2} y_1 x_2 \operatorname{Sin}\omega = \tfrac{1}{2}\operatorname{Sin}\omega\,.\,(x_1 y_2 - y_1 x_2),$$

in either figure; for $-y_1x_2 =$ a positive area in the second, x_2 being negative and y_1 positive, [3].

[51] Sin ώ (xў lĕ ýx) at ŭn twó, pron zy le wix at untó.
Is twíce Are (Or′ un twó). un for unity.

i.e. $\text{Sin}\,\omega \times (x_1y_2 - y_1x_2)$, *xy at* 1, 2, *less yx at* 1, 2, is *twice* the *area* included within the points *Origin*, (x_1, y_1) and (x_2, y_2): ω being the angle between the co-ordinate axes. This is true for *every* pair (x_1, y_1), (x_2, y_2).

The area may have either sign, as it is considered to be space from Oq to Op, in the positive direction from OX towards OY, or space from Op to Oq, in the opposite direction from OY towards OX; and these two spaces, *if* the sign is to be taken into account at all, *must* have opposite signs. Look now at any triangle 1 2 3; we see that, if $O1\,2$ is the $\triangle(O12)$ &c.,



$$\triangle 123 = O12 - (O13 + O32),$$

if we consider $O12$, $O32$, and $O13$, to be spaces described by angular motion from OX towards OY. Then, if 1, 2, and 3, be (x_1y_1), (x_2y_2), and (x_3y_3),

$$\triangle(123) = \tfrac{1}{2}\,\text{Sin}\,\omega\{x_1y_2 - y_1x_2 - (x_3y_2 - y_3x_2) - (x_1y_3 - y_1x_3)\} \text{ by [51], or}$$

$$\triangle(123) = \tfrac{1}{2}\,\text{Sin}\,\omega\{x_1.(y_2 - y_3) + x_2.(y_3 - y_1) + x_3.(y_1 - y_2)\}.$$

This then is the area of *any* triangle, whose angular points are $(x = x_1,\ y = y_1)$, $(x = x_2,\ y = y_2)$, $(x = x_3,\ y = y_3)$; provided that in this expression you take care to put for x_1, &c. the proper numbers with the proper signs. Can you remember this?

Richard:—Why it is exactly 'ter xDi.y's is nil,' if you put 0 for 3; just like the equation to a given line through 1 and 2. [9].

Uncle Pen.:—It is not 'nil' however; for it is

$$x_1.(y_2 - y_3) + x_2.(y_3 - y_1) + x_3.(y_1 - y_2) = \frac{2\,.\,\text{area}\,(123)}{\text{Sin}\,\omega}, \quad \text{(B)}$$

which is never nil, unless the points 123 are in the same line. Let 3 fall anywhere in the line (12), and then it becomes

$$x_0.(y_1 - y_2) + x_1.(y_2 - y_0) + x_2.(y_0 - y_1) = 0 = 2\,\text{area}\,(012)\text{:Sin}\,\omega,$$

$x_0 y_0$ being any point on that line. This merely affirms that the area of the triangle whose angles are at $(x_0 y_0)$, $(x_1 y_1)$, and $(x_2 y_2) = 0$.

Jane:—How elegantly does this interpret the formula, 'ter xDiy's is nil!'

Having fixed the algebraical meaning of $O12$ and $O21$ to be $\frac{1}{2}$ Sin $\omega\,(x_1 y_2 - y_1 x_2)$, and $\frac{1}{2}$ Sin $\omega\,(x_2 y_1 - y_2 x_1)$, it is evidently true that

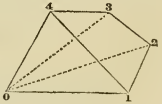
$$O12 = -O21, \text{ and } O12 + O21 = 0 = \text{no area.}$$

We may give to these equations a congruous geometrical meaning, and consider them to be brief methods of affirming the obvious truth, that the angular description of the area $O12$ in the last figure, by a line through O which sweeps from 1 to 2, while its extremity travels along the line 12, and then sweeps in the same way from 2 to 1, has done and undone the same thing, and has traced out no area, the two spaces described having destroyed each other. It is as though the line had not moved at all. We may write the triangle 123 thus:

$$\triangle\, 123 = O12 + O23 + O31,$$

and give to the right member either its algebraic or its geometrical meaning, including this conception of motion.

In the same way the area of the quadrilateral (1234) made by joining the points 1234 in *that order*, may be symbolized thus:



$$(1234) = O12 + O23 + O34 + O41\,;$$

for it is plainly the sum of the triangles $O12 + O23 + O34$, less the triangle $O41$, i. e. + the triangle $O14$; since $O41 = -O14$. In algebraic language,

Area (1234)

$$= \tfrac{1}{2}\{(x_1y_2 - y_1x_2) + (x_2y_3 - y_2x_3) + (x_3y_4 - y_3x_4) + (x_4y_1 - y_4x_1)\}\,\text{Sin}\,\omega,$$

and this certainly true, whatever be the points $x_1 y_1$ &c., or the co-ordinate axes employed, if in this expression the proper values and signs be put for the co-ordinates. The same thing is true of the area of any polygon $123456 \ldots n$ made by joining n points *in that order*,

$$\text{Area of polygon} = O12 + O23 + O34 + \ldots + O(n-1)\,n + On\,1,$$

where $O(n-1)\,n = \frac{1}{2}$ Sin $\omega\,(x_{n-1} y_n - y_{n-1} x_n)$, in which for x_n &c.

are to put their proper values with their proper signs. All this is true, even of polygons whose sides drawn in that order cross each other (not produced); but you had better not trouble yourself with such figures at present.

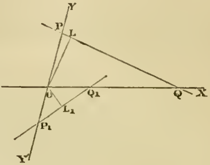
53. Let $Ax + By = C$,

or $\frac{x}{C:A} + \frac{y}{C:B} = 1$, v. (9),

be the equation of any line PQ,

and $A_1x - B_1y = C_1$,

or $\frac{x}{C_1:A_1} - \frac{y}{C_1:B} = 1$,



that of a line P_1Q_1, the two lines subtending supplemental angles at O, and let the co-ordinate angle $POQ = \omega$. It is required to find from these equations the length (disregarding the signs) of the perpendiculars OL and OL_1 on these lines from the origin. By Prop. (B 20) and [32], we know that

$$OL \,.\, QP = \text{Sin}\,\omega \,.\, OP \,.\, OQ = 2\,(\text{Area } OPQ),$$

$$OL_1 \,.\, Q_1P_1 = \text{Sin}\,(\pi - \omega)\, OP_1 \,.\, OQ_1 = 2\,(\text{Area } OP_1Q_1);$$

whence by division of equals,

$$OL = \text{Sin}\,\omega \,.\, \frac{OP \,.\, OQ}{QP};$$

and $OL_1 = \text{Sin}\,(\pi - \omega)\,\frac{OP_1 \,.\, OQ_1}{Q_1P_1}$; i. e., [26 B],

$$OL = \text{Sin}\,\omega \,.\, \frac{OP.\ OQ}{\pm(OP^2 + OQ^2 - 2OP \,.\, OQ\,\text{Cos}\,.\,\omega)^{\frac{1}{2}}};$$

and

$$OL_1 = \text{Sin}\,(\pi - \omega)\,\frac{OP_1 \,.\, OQ_1}{\pm\{OP_1^2 + OQ_1^2 - 2OP_1 \,.\, OQ_1\,\text{Cos}\,(\pi - \omega)\}^{\frac{1}{2}}}$$

Observe, that in [26 B] a and c represent two numbers of the same sign, and consequently OP_1 and OQ_1 must be considered as of the same sign; to find Q_1P_1, we have no account to take of their signs, but of their *lengths* only, which are by [5′] $C_1:B_1$ and $C_1:A_1$. Then

$$OL = \frac{\operatorname{Sin}\omega \cdot \frac{C}{B} \cdot \frac{C}{A}}{\pm\left(\frac{C^2}{B^2} + \frac{C^2}{A^2} - 2\frac{C}{B}\cdot\frac{C}{A}\operatorname{Cos}\omega\right)^{\frac{1}{2}}},$$

$$\text{and } OL_1 = \frac{\operatorname{Sin}(\pi - \omega) \cdot \frac{C_1}{B_1} \cdot \frac{C_1}{A_1}}{\pm\left\{\frac{C_1^2}{B_1^2} + \frac{C_1^2}{A_1^2} - \frac{2C_1}{B_1}\cdot\frac{C_1}{A_1}\operatorname{Cos}(\pi - \omega)\right\}^{\frac{1}{2}}}.$$

The numerator and denominator of this latter fraction being both multiplied by $\frac{B_1A_1}{C_1}$ or its equal $\left(\frac{B_1^2A_1^2}{C_1^2}\right)^{\frac{1}{2}}$, we obtain, [23],

$$OL_1 = \frac{\pm C_1 \operatorname{Sin}\omega}{\left(\frac{B_1^2A_1^2}{C_1^2}\right)^{\frac{1}{2}} \cdot \left(\frac{C_1^2}{B_1^2} + \frac{C_1^2}{A_1^2} + \frac{2C_1^2}{B_1A_1}\operatorname{Cos}\omega\right)^{\frac{1}{2}}};$$

or by [47] 'Prod. roots, &c.,'

$$OL_1 = \frac{\pm C_1 \operatorname{Sin}\omega}{(A_1^2 + B_1^2 + 2A_1B_1\operatorname{Cos}\omega)^{\frac{1}{2}}};$$

$$\text{for } \frac{B_1^2A_1^2}{C_1^2} \cdot \frac{C_1^2}{B_1^2} = A_1^2; \text{ \&c.,}$$

$$\text{and } OL = \frac{\pm C \operatorname{Sin}\omega}{(A^2 + B^2 - 2AB\operatorname{Cos}\omega)^{\frac{1}{2}}}.$$

When $\omega = 90^\circ$, this is

$$OL_1 = \frac{\pm C_1}{(A_1^2 + B_1^2)^{\frac{1}{2}}}, \quad OL = \frac{\pm C}{(A^2 + B^2)^{\frac{1}{2}}}.$$

The denominators of OL and OL_1 are exactly of the same form, if you take into account that B_1 is negative, while B is positive: so that if $ax + by = c$ be *any* line, referred to these axes, the perpendicular on it from the origin

$$= \pm c \operatorname{Sin}\omega : (a^2 + b^2 - 2ab\operatorname{Cos}\omega)^{\frac{1}{2}}.$$

For example, the perpendicular on $y + 3x = 4$ is

$$\pm 4 \operatorname{Sin}\omega : (1^2 + 9 - 6\operatorname{Cos}\omega)^{\frac{1}{2}},$$

and the perpendicular on $y - 3x = 4$ is

$$\pm 4 \operatorname{Sin}\omega : (1^2 + 9 + 6\operatorname{Cos}\omega)^{\frac{1}{2}}.$$

We can in general take our choice of signs, as in $\pm\sqrt{2}$; and may consider the perpendicular positive, so long as we contradict no previous supposition thereby.

Let us for shortness call the perpendicular on any line from the origin the *Lor* of that line, distance of *L*ine from *O*rigin. And let us call *the expression* $(a^2+b^2-2ab \text{ Cos } \omega)^{\frac{1}{2}}$, *bas*$(ab\omega)$, (pron. *ăbó;*): it represents, *when* a *and* b *are numbers of the same sign,* the *base* of the triangle [26 B] whose sides are a and b containing the angle ω. If a and b in *bas*$(ab\omega)$ are numbers of unlike signs, this geometrical explanation fails; but, from what precedes, you can see that

$$(a^2+b^2+2ab \text{ Cos } \omega)^{\frac{1}{2}}$$

is the base of the triangle whose sides are $+a$ and $+b$, containing the angle $(\pi-\omega)$: in fact it is

$$\{a^2+b^2-2ab \text{ Cos } (\pi-\omega)\}^{\frac{1}{2}}.$$

Let us call A and B, in $Ax+By-C=0$, the *fics*, i. e. the coef*fic*ient*s* of x and y: then bas(fics ω) will stand for the expression $(A^2+B^2-2AB \text{ Cos } \omega)^{\frac{1}{2}}$, whether the *fics* have like or unlike signs, i. e. whether $-2AB$ represents a negative or positive number. You may say

[52] Lŏr's Cón. Sin ω by bás(fics ω):
Is Cón. cīpŏth fics, if right is ω.

ci = cip, v. [44], poth vid. [12.]

i.e. the *Lor* = (the *con*stant term of the equation) × sin ω, divided *by* bas (fics ω) explained above; and = when ω is 90^0 *Con*stant × the reciprocal of the *poth* (21) of the coef*fic*ient*s*. You will find the value of these *abbreviations* as we proceed.

The two lines $y+3x=\pm 4$, have *Lors* of the same *length*, and the pair $y-3x=\pm 4$, have equal *Lors* also.

If the equation of the first line were written

$$\frac{y}{4}+\frac{x}{4:3}=1;$$

the *Lor* would be

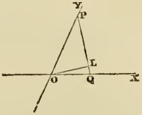
$$\text{Sin } \omega:\left(\frac{1}{16}+\frac{9}{16}-\frac{6}{16}\text{ Cos } \omega\right)^{\frac{1}{2}}, \text{ or } 1:\sqrt{\frac{10}{16}}, \text{ when } \omega=90.$$

You see readily that this is the same value as before, p. 129.

54. Let $ax + by \pm c$ be the equation of any line referred to any axes OX OY, and let l be the *Lor* of this line, or perpendicular on it from the origin. It is evident that $max + mby = mc$ is the same line still; for every value of x and y that satisfies the first will satisfy the second equation. Let $m = l{:}c$, then $mc = l$; and

$$max + mby = l, \text{ or } a_1x + b_1y = l, \text{ (if } ma = a_1 \text{ \&c.)},$$

is the equation to the same line. What is the geometrical meaning of a_1 and b_1 considering their *values* only and not their signs? Suppose PQ to be the line; when $x = 0$, we have



$$b_1y = l, \text{ or } b_1 = l{:}y = OL{:}OP,$$

$$\text{i. e. } b_1 = \text{Sin } OPQ;$$

and when $y = 0$, we have

$$a_1x = l; \ a_1 = l{:}x = OL{:}OQ = \text{Sin } OQP.$$

The equation then is

$$\text{Sin } OQP . x + \text{Sin } OPQ . y = l; \qquad 1.$$

whence it appears that if the absolute term of the equation to a line is the *Lor* or $\perp$ from the origin, the coefficients of x and y, considering their numerical values apart from their signs, are the sines of the angles which the line makes with OX and with OY. And you may easily prove also, by writing the equation in the form

$$\frac{x}{l{:}\text{Sin } OQP} + \frac{y}{l{:}\text{Sin } OPQ} = 1, \text{ as in [9]},$$

that if the coefficients are numerically the sines of these angles, its absolute term is exactly the perpendicular from the origin.

Let then $ax + by - l = 0$ be a line PQ whose *Lor* is l. We know at once what the numbers a and b are. The equation is not true for a point (x_1y_1) *not* in the line;

$$ax_1 + by_1 - l \text{ is } \textit{not} = 0, \text{ but}$$

$$ax_1 + by_1 - l = \lambda, \text{ or } ax_1 + by_1 - l - \lambda = 0,$$

is the truth, λ being some number positive or negative. Yet the equation $ax + by - l - \lambda = 0$ represents *some* line pq of which $(x_1 \ y_1)$ is a point, because its equation is true at

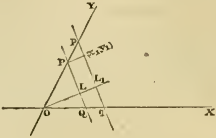
$(x_1\, y_1)$; and by [5] we know that this line is parallel to $ax + by - l = 0$, both lines being parallel to $ax + by = 0$. Hence both lines make the same angles with OX and OY, and the coefficients a and b of the variables in each line are as just pointed out, numerically the sines of these angles; therefore

$$ax + by = l + \lambda$$

is a line whose *Lor* is

$$l + \lambda: \text{ i.e. } OL_1 = OL + \lambda,$$

and $\lambda = OL - OL = LL_1$.



Thus we find that

$$ax_1 + by_1 - l = \lambda = LL_1,$$

and this is exactly the perpendicular distance between the lines PQ and pq: in other words, $ax_1 + by_1 - l =$ the perpendicular from $(x_1\, y_1)$ [2] upon the line PQ, which has for its equation $ax + by - l = 0$.

Thus we have proved that if in the equation

L. $\quad ax + by - l = 0 = \text{u}$, u standing for $(ax + by - l)$,

l is the perpendicular from the origin, a and b being the sines of the angle between the line and the axes OX and OY, the values of u for different co-ordinates $(x_1\, y_1)$ $(x_2\, y_2)$ &c., which values we may call u_1, u_2, &c., u_1 being $(ax_1 + by_1 - l)$ &c., are exactly the lengths of the perpendiculars, from those points respectively, on the line $\text{u} = 0$. This affirmation $\text{u} = 0$ is true of all points in the line $\text{u} = 0$, i.e. the perpendiculars from any point of this line upon this line $= 0$. If a line be given by any equation

$$y + cx - b = 0 = u,$$

in which b is or is not the *Lor* or perpendicular from the origin, we obtain $\text{u} = 0$ of the form L, by multiplying u by $l{:}b$, l being the *Lor;* so that $(l{:}b)\, u = \text{u}$, as at the beginning of this article, when $m = l{:}c$.

[53] vălȣ′. vĭ LŏrCón is pér lĭn. ŭpón.

i.e. if $u = 0 = ȣ(\delta\epsilon\nu)$ be the equation $(ax + by - c = 0)$ of any given line; val ȣ, the *val*ue of that $ȣ(\delta\epsilon\nu)$ (made by putting any $(x_1 y_1)$ for the co-ordinates,) × the quotient Lor:Con, (*vi LorCon*), is the *per*pendicular *upon* the *l*ine from the point $(x_1 y_1)$. Con means the constant or absolute term.

Jane:—What is this ȣδεν?

Richard:—It is Greek for *nothing;* you are to pronounce owden; ȣ is ou.

Uncle Pen.:—Find now the distance of the point

$$(x_1 = 2,\ y_1 = 7) \text{ from the line } 3y - 5x + 4 = 0 = u,$$

referred to axes inclined at the angle $\omega = 30^\circ$.

Richard:—By [53] the perpendicular is

$$(3y_1 - 5x_1 + 4) . (l{:}4), \text{ if } l \text{ be the } Lor;$$

and by [52],

$$\frac{l}{4} = \frac{\text{Sin } 30}{\pm\sqrt{3^2 + 5^2 + 30 \text{ Cos } 30^0}}.$$

What now is Sin 30?

Jane:—Half the angle of an equilateral is 30^0; and besides, Sin 30 must be Cos 60 by [23, G]; and this we have seen (51, Ex. 1) to be $\frac{1}{2}$. Hence by [25], Cos $30 = + \frac{\sqrt{3}}{2}$.

The expression for the perpendicular is

$$(3y_1 - 5x_1 + 4) . \frac{\frac{1}{2}}{(9 + 25 + 15\sqrt{3})^{\frac{1}{2}}}$$

$$= (21 - 10 + 4) . \frac{1}{2\sqrt{34 + 15\sqrt{3}}}$$

$$= \frac{7{\cdot}5}{\pm\sqrt{34 + 15\sqrt{3}}}.$$

Uncle Pen.:—Find now the distance of the point

$$(x_1 = -3,\ y_1 = 5) \text{ from the line } 8x + 7y = 0,$$

referred to right axes.

Richard:—This will be simpler, I fancy: but there is no Con! and of course no *Lor*, the line passing through the origin. Then vi. *Lor Con* is $\frac{0}{0}$. I give it up.

Uncle Pen.:—Proceed boldly to find Lor by [52].

Richard:—Am I then to say

$$0 = \frac{0}{\pm\sqrt{8^2 + 7^2}}, \text{ and } \frac{0}{0} = \frac{1}{\pm\sqrt{8^2 + 7^2}}?$$

Uncle Pen.:—You can safely say, putting l and c for *Lor* and *Con*,

$$l = \frac{c}{\sqrt{8^2+7^2}}, \text{ and } \frac{l}{c} = \frac{1}{\sqrt{8^2+7^2}},$$

of any line $8x+7y=c$; and this is true for all values positive and negative of c, each different c giving of course a different line and *Lor*. It is true also when $c=0$; for we know that the value of l vanishes when $c=0$, i.e. when the line passes through the origin. Here then is a case of $\frac{0}{0}$, arising from the unlimited diminution of both numerator and denominator of a fraction, the value of the fraction continuing still unchanged. The perpendicular required is certainly

$$\frac{8 \times -3 + 7 \times 5}{\pm\sqrt{64+49}} = \frac{11}{\sqrt{113}}:$$

and this you will find to be the *Lor* of the line,

$$8x + 7y = -8 \times 3 + 7 \times 5,$$

which passes through $(-3, 5)$, and is parallel to the given line through the origin. This *Lor* and the perpendicular are equal, being opposite sides of a rectangle, of which the two parallel lines form the other sides.

Generally the perpendicular from

$$x_1y_1 \text{ on } ax+by-c=0 \text{ is}$$

P. $$(ax_1+by_1-c)\frac{\text{Sin } \omega}{(a^2+b^2-2ab \text{ Cos } \omega)^{\frac{1}{2}}}.$$

The abbreviation *bas* is founded on 'sq.b &c.,' [26], and always has the same import, as to signs of the algebraical symbols, whatever shape it may assume after substitution of arithmetical values.

The following observation will presently be useful. If we have the given lines

$$u = 0 = ax+by+c, \qquad v = 0 = a_1x + b_1y + c_1,$$
$$w = 0 = a_2x + b_2y + c_2, \qquad z = 0 = a_3x, \text{ \&c.},$$

and l, m, n, be known numbers; then

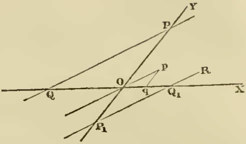
$$u + lv + mw + nz = 0,$$

is also a given line; for it is

$$(a + la_1 + ma_2 + na_3)\,x + (b + lb_1 + mb_2 + nb_3)$$
$$y + (c + lc_1 + mc_2 + nc_3) = 0.$$

LESSON XVII.

55. THE geometrical meaning of e in $y - b = ex$, when y has the coefficient unity, and is on the opposite side of the equation from ex, is very important. Take [5] the parallel line $y = ex$, or Op. If pq be the ordinate at p,



$pq{:}Oq = e$, or by [34],

$$\frac{\operatorname{Sin} pOq}{\operatorname{Sin} Opq} = e = \frac{\operatorname{Sin} pOq}{\operatorname{Sin} YOp}; \quad \text{'alter. ins. =,' [1].}$$

Let the angle between *any* line $Ax + By - C = 0$, and OX, on the *upper* side towards the *right*, be called β; then is $YOp = \omega - \beta$, pOq being the β of Op:

(a) $$\frac{\operatorname{Sin} \beta}{\operatorname{Sin} (\omega - \beta)} = e = \frac{\operatorname{Sin} pOq}{\operatorname{Sin}(YOX - pOq)},$$

or by [27],

$$\operatorname{Sin} \beta = e\,(\operatorname{Sin} \omega \operatorname{Cos} \beta - \operatorname{Cos} \omega \operatorname{Sin} \beta),$$

whence, transposing,

$$\operatorname{Sin} \beta + e \operatorname{Cos} \omega \operatorname{Sin} \beta = e \operatorname{Sin} \omega \operatorname{Cos} \beta,$$

and by division by $\cos \beta$, [22.]

$$\tan \beta + e \operatorname{Cos} \omega \tan \beta = e \operatorname{Sin} \omega; \quad \text{then dividing by } 1 + e \cos \omega,$$

(b) $$\tan \beta = \frac{e \sin \omega}{1 + e \operatorname{Cos} \omega}: \text{ this is, by [25],}$$

(b′) $$\tan \beta = e, \text{ if } \omega = \tfrac{1}{2}\pi,$$

the case of right axes.

All lines parallel to $y = ex$ have the same β, as

$$\beta = pOq = PQO = RQ_1X;$$

and $\operatorname{Sin} \beta$ and $\operatorname{Sin} (\omega - \beta)$ are the same for all, consequently e has the same meaning and value, $\operatorname{Sin} \beta{:}\operatorname{Sin} (\omega - \beta)$, in them all.

You may pronounce β like b, and say,

[54] (a) Síβ to SĭD(ωβ), or táβ in RAx,
pron. Dωβ, dōbe; to for (:).
Is é in ў's éx : β ŭp ríght LĭnAx. 's for =.
(b) tăβ's é Sin ώ bў S(ŭn é Cos ώ). vid. D S [14].

The ratio of *sin* β *to Sin* (*Diff.* of $\omega\beta$), sin β:sin $(\omega-\beta)$, which is *tan* β in *Right Axes*, is e, in $y=ex$: β being always *up*per angle on the *right* between the *Line* and the *Axis* of x. For all axes, $ta\beta = e \sin\omega$ *by Sum* (*unity* and $e \cos \omega$).

It is very easy now *to find the angle between any two lines* $y=ex+b$ and $y=e_1x+b_1$, the axes and their angle ω being given.

Let them be PQ and PQ_1, meeting each other in P in the positive angle, and OX in Q and Q_1; draw the figure, and suppose $PQ_1X > PQX$. Then by Prop. D, $PQ_1X = QPQ_1 + PQX$, $PQ_1X - PQX = \beta_1 - \beta = QPQ_1$ the angle between the lines.

By [36] $\tan(\beta_1-\beta) = \dfrac{\tan\beta_1 - \tan\beta}{1+\tan\beta_1\tan\beta}$, or by [54, b].

$$\text{(c)}\quad \tan QPQ_1 = \frac{\dfrac{e_1 \operatorname{Sin}\omega}{1+e_1\operatorname{Cos}\omega} - \dfrac{e\operatorname{Sin}\omega}{1+e\operatorname{Cos}\omega}}{1+\dfrac{e_1\operatorname{Sin}\omega}{1+e_1\operatorname{Cos}\omega}\cdot\dfrac{e\operatorname{Sin}\omega}{1+e\operatorname{Cos}\omega}} = \tan(\beta_1-\beta).$$

When $\omega=\frac{1}{2}\pi$, this is simply, [25],

$$\text{(d)}\qquad \tan QPQ_1 = \frac{e_1-e}{1+ee_1}; \quad = \tan(\beta_1-\beta).$$

You may effect for yourselves the reduction of (c) to

$$\text{(c}')\qquad \tan QPQ_1 = \frac{(e_1-e)\operatorname{Sin}\omega}{1+(e+e_1)\operatorname{Cos}\omega+ee_1} = \tan(\beta_1-\beta).$$

This gives the angle between two given lines for any axes right or oblique: the distance between two points you know for the case of right axes by [12]. Try for yourselves to prove that when ω is oblique, the distance r between the point (l, a) and (x, y) is always

$$r = \{(x-l)^2+(a-y)^2-2.(x-l).(a-y)\operatorname{Cos}\omega\}^{\frac{1}{2}}, \qquad \text{e.}$$

or = the base of the $\triangle$ whose sides, *when both of one sign*, are $(x-l)$ and $(a-y)$, and the vertical angle ω, *not* $(\pi-\omega)$. Call the angle $(\beta_1-\beta)$, LiL Diβs, *Diff* of β's, angle between *Line* and *Line*, and say:

[55] (e) băs[Dí(xl) ω D(áy)] joins xy̆ to lá,

D = Diff. xy a dissyl.

(c) LiL Díβ's you knów by ʻtăn(á mol ώ), [36].

Bas here as in [52]; Di(xl) and D(ay) are the sides containing the angle ω, and this *bas* becomes the *poth* of [12] when $\omega = 90$. The second line, together with Prop. D [36] and [54 b] will always put you in possession of $\tan(\beta_1 - \beta)$. $\operatorname{Cos}\omega$ changes sign with $(x-l).(a-y)$. Draw this.

56. If $y = ex + b$, and $y = e_1 x + b_1$, are lines at right angles to each other, $\beta - \beta_1 = \frac{1}{2}\pi$, or else $\beta_1 - \beta = \frac{1}{2}\pi$, whence either $\beta_1 = \frac{1}{2}\pi + \beta$, or $\beta_1 = \beta - \frac{1}{2}\pi$. In the first case,

$$\tan\beta_1 = \tan(\beta + \tfrac{1}{2}\pi) = \frac{\operatorname{Sin}(\beta + \frac{1}{2}\pi)}{\operatorname{Cos}(\beta + \frac{1}{2}\pi)}$$

$$= [23\ G], \frac{\operatorname{Cos}(-\beta)}{\operatorname{Sin}(-\beta)} = \frac{\operatorname{Cos}\beta}{-\operatorname{Sin}\beta} = \frac{-1}{\tan\beta} = -\cot\beta;$$

in the other case,

$$\tan\beta_1 = \tan(\beta - \tfrac{1}{2}\pi) = \frac{-\operatorname{Sin}(\frac{1}{2}\pi - \beta)}{\operatorname{Cos}(\frac{1}{2}\pi - \beta)} = -\frac{1}{\tan\beta} = -\cot\beta;$$

so that, in either case,

$$\tan\beta_1 = -\cot\beta, \text{ and } \cot\beta_1 = -\tan\beta, \qquad \text{(a)}$$

$$\text{giving, by [54, b], } \frac{e\operatorname{Sin}\omega}{1 + e\operatorname{Cos}\omega} = -\frac{1 + e_1\operatorname{Cos}\omega}{e_1\operatorname{Sin}\omega}, \qquad \text{(b)}$$

or, mul. both sides by $(1 + e\operatorname{Cos}\omega)(e_1\operatorname{Sin}\omega)$,

$$ee_1\operatorname{Sin}^2\omega = -1 - e\operatorname{Cos}\omega - e_1\operatorname{Cos}\omega - ee_1\operatorname{Cos}^2\omega,$$

whence by transposition and [25],

$e_1(e + \operatorname{Cos}\omega) = -(1 + e\operatorname{Cos}\omega)$, and then by division,

$$e_1 = -\frac{1 + e\operatorname{Cos}\omega}{e + \operatorname{Cos}\omega}; \qquad \text{(c)}$$

which becomes $e_1 = -\dfrac{1}{e}$, for right co-ordinates.

To remember (a) (b) (c), say, thinking of [54, b],

[56] gĭl's lé cotá.β is pérlĭn's taβ.

gil is *g*iven *l*ine; a given line's negative cota β (*le* = − or *neg.*) is the *tan* β of the *line per*pendicular to it. Thus β_1 and e_1 are found by β and e.

Let now the sides $OB = a$, and $OA = b$, of any triangle, be taken for the positive axes of y and x. If BP and AR

be perpendiculars from B and A, on OA and OB, and $\angle BOA = \omega$,

$$\frac{y}{a} + \frac{x}{a \operatorname{Cos} \omega} = 1, \text{ is } BP, [5'] [22];$$

$$\text{and } \frac{y}{b \operatorname{Cos} \omega} + \frac{x}{b} = 1, \text{ is } AR.$$

When x and y are the same in both these, by subtraction

$$y\left(\frac{1}{a} - \frac{1}{b \operatorname{Cos} \omega}\right) + x \cdot \left(\frac{1}{a \operatorname{Cos} \omega} - \frac{1}{b}\right) = 0, \text{ is true, or,}$$

multiplying both sides of this equation by $ab \operatorname{Cos} \omega$, the line

$$y \cdot (b \operatorname{Cos} \omega - a) + x\,(b - a \operatorname{Cos} \omega) = 0, \qquad \text{(p)}$$

contains the intersection of AR and BP.

The equation to BA is [5′]

$$\frac{y}{a} + \frac{x}{b} = 1, \text{ or } y = -\frac{a}{b}x + a:$$

the perpendicular on this line from O is of the form $y = e_1 x$, [4], i. e.

$$y = -\frac{1 - (a{:}b) \operatorname{Cos} \omega}{-a{:}b + \operatorname{Cos} \omega}\, x, \text{ by [56] and [54, b], } (e \text{ being } -a{:}b)$$

which you can easily prove to be none other than the line (p), through the intersection of AR and BP. We have proved that *the three perpendiculars from the angles of any triangle let fall on the opposite sides meet in a point.*

57. *Jane:*—You have shewn us how to find 'perc bic. and biCang' in any given triangle, and the equations of these lines can be formed by [9], if we have two points known in each. Suppose that a triangle is given, can you find from its three angular points the equations of these three lines? We can find the sides and angles.

Uncle Pen.:—Let us try what we can make of the problem: *the co-ordinates of any three points being given, determining a triangle, to find thereby the equations to the perpendicular from* C *on* c, *the bisector of* c *from* C, *and the bisector of the angle* C *in the triangle.* We can write down the equations of the sides, having two points in each, by [9] whatever be the axes and origin chosen: we can find by [52] the Lor of each line, and reduce the equations of the

three sides to a = 0, b = 0, c = 0, of the form (54, L). What meaning do you see, Jane, in this symbol a?

Jane:—I see three constant numbers, two of them sines, and one of them a *Lor:* I see my friends x and y amusing themselves as usual: so long as a = 0 is true, they drag each other along the side a of the triangle; and when a *not* = 0, a is always the perpendicular on that side a from the point (xy) exhibited.

Uncle Pen.:—If I now say, without taking the trouble to form these equations actually, let $\mathrm{a} - \phi\mathrm{b} = 0 = \gamma$ be the equation of *Ce*, (Art. 30), the bisector of the angle C, what do you gather, Richard, from this assertion, $\mathrm{a} - \phi\mathrm{b} = 0$?

Richard:—I understand that a certain perpendicular on the side a is ϕ times a certain perpendicular on the side b.

Jane:—But, surely, the two perpendiculars are drawn from the same point (xy).

Richard:—How does that appear in the assertion? I stand by what is written. The points in a = 0 have no connection with those in b = 0: how does it appear that the same x and y *must* stand in a as in b?

Uncle Pen.:—Plainly from what I laid down; $\gamma = 0$ is to be considered the equation to a line, and cannot contain two values of x at the same moment, or two values of y.

Richard:—The assertion then simply is, that from every point of the line $\gamma = 0$, the perpendicular on a is ϕ times that on b; but what is ϕ?

Uncle Pen.:—It is intended for a constant, whose value we have yet to find: every different ϕ gives a different line; but, whatever ϕ may be, the line will pass through C the intersection of a = 0 and b = 0, for $\gamma = 0$ must be true for the x and y which make these two each = 0. There are as many values of ϕ assignable, as of lines through C, i. e. an infinite number, and thus we shall have the equation of the line *Ce* before us, if we can determine the proper ϕ. This is easily done, because ϕ is constant, and the same for every point in $\gamma = 0$. If you draw the perpendiculars from e on the sides a and b, you see by [22 H] that they are equal, C being bisected: that is, a = b or a − b = 0 for the x and y of e. Thus ϕ is found to be unity, and $\mathrm{a} - \mathrm{b} = 0 = \gamma$ is the equation of the line *Ce*, being true at two points of it, C and e.

We seek next the equation to Cd the perpendicular on c, and it is evident that $\mathrm{a} - \phi_1 \mathrm{b} = 0 = P_c$ is this equation, if ϕ_1 can be properly determined. This asserts that $\mathrm{a}:\mathrm{b} = \phi_1$, i. e. the ratio of the perpendiculars from every point of $P = 0$ on the sides a and b is the number ϕ_1. Draw these perpendiculars from d: one is $Cd \cdot \operatorname{Sin} BCd$ or $Cd \cdot \operatorname{Cos} B$ by [23, G], and the other is $Cd \cdot \operatorname{Cos} A$, the ratio of which two products is $\operatorname{Cos} B : \operatorname{Cos} A$. This is ϕ_1 at d, and therefore at every point of $P_c = 0$; hence

$$\mathrm{a} - \frac{\operatorname{Cos} B}{\operatorname{Cos} A}\mathrm{b} = 0; \quad P_c = \mathrm{a} \operatorname{Cos} A - \mathrm{b} \operatorname{Cos} B = 0,$$

both sides being multiplied by $\operatorname{Cos} A$, is the *perc* required.

You can easily prove from [22 H] that the ratio of the perpendiculars let fall from f on a and b is $\operatorname{Sin} B : \operatorname{Sin} A$; hence

$$\mathrm{a} \operatorname{Sin} A - \mathrm{b} \operatorname{Sin} B = 0 = Q_c$$

is the bisector of c, being true at C and at f, two points of *bic*. It will be very profitable for you to remember and to meditate these results: you can say to yourself

[58] a Cŏs A'ng's b Cŏ.Bang,
a Sin Ang's b SiBang,
b's a, (in Rom ba) ăre pĕrc bíc ănd biCang. vid. [18].

i.e. $\mathrm{a} \cos A = \mathrm{b} \cos B$, $\mathrm{a} \sin A = \mathrm{b} \sin B$, $\mathrm{b} = \mathrm{a}$, the symbols b and a being *Rom*an letters, denoting the form (54 L), are the equations of the perpendicular on c, the bisector of c, and bisector of C.

58. By going round the circle *abca*... we obtain

$\gamma = \mathrm{a} - \mathrm{b} = 0$	$\mathrm{a} \operatorname{Cos} A - \mathrm{b} \operatorname{Cos} B = 0 = P_c$	$\mathrm{a} \operatorname{Sin} A - \mathrm{b} \operatorname{Sin} B = 0 = Q_c$
$\alpha = \mathrm{b} - \mathrm{c} = 0$	$\mathrm{b} \operatorname{Cos} B - \mathrm{c} \operatorname{Cos} C = 0 = P_a$	$\mathrm{b} \operatorname{Sin} B - \mathrm{c} \operatorname{Sin} C = 0 = Q_a$
$\beta = \mathrm{c} - \mathrm{a} = 0$	$\mathrm{c} \operatorname{Cos} C - \mathrm{a} \operatorname{Cos} A = 0 = P_b$	$\mathrm{c} \operatorname{Sin} C - \mathrm{a} \operatorname{Sin} A = 0 = Q_b$

for the three bisectors of the angles, the three perpendiculars on the sides, and the three bisectors of the sides. When γ and α are both $= 0$, which can only be at the point of their intersection, you see by addition, that $\gamma + \alpha = 0 = \beta = \mathrm{a} - \mathrm{c}$; for b in γ and b in α are the same number for the (xy) of intersection. This point is thus also in $\beta = 0$, or *the three bisectors α, β, γ, as in* (43), *meet in a point*. By the same argument $P_b = 0$ when $P_c = P_a = 0$, and the *three perpendiculars on the sides meet in a point*, as already proved above. Also $Q_b = 0$, if both Q_b and $Q_a = 0$, *or the three bisectors of*

the sides meet in a point. In this article and (43) you have proof of this proposition:

In any triangle, the three perpendiculars at the centres of the sides meet in a point, as do also the three bisectors of the angles, the three bisectors of the sides from the opposite angles, and the three perpendiculars on the sides from the angles.

Let BPB', APA', CPC' be lines drawn from the angles of a triangle ABC through any point P without or within the triangle to meet the sides in $A'B'C'$. Draw two figures.

By [34] we have in the triangle $AC'P$, &c.,

$AC' = \text{Sin}\,APC'.PA : \text{Sin}\,AC'C$	$AB' = \text{Sin}\,APB'.PA : \text{Sin}\,AB'B$
$BA' = \text{Sin}\,BPA'.PB : \text{Sin}\,BA'A$	$BC' = \text{Sin}\,BPC'.PB : \text{Sin}\,BC'C$
$CB' = \text{Sin}\,CPB'.PC : \text{Sin}\,CB'B$	$CA' = \text{Sin}\,CPA'PC : \text{Sin}\,CA'A$;

whence it follows that

$$AB'.BC'.CA' = AC'.CB'.BA',$$

if you put for $\text{Sin}\,APC'$ its equal $\text{Sin}\,CPA'$, for $\text{Sin}\,AC'C$ its equal [23 L] $\text{Sin}\,BC'C$, and so on. Further we have [34],

$\text{Sin}\,ACC' = AC'.\text{Sin}\,A : CC'$	$\text{Sin}\,ABB' = AB'.\text{Sin}\,A : BB'$
$\text{Sin}\,BA'A = BA'.\text{Sin}\,B : AA'$	$\text{Sin}\,BCC' = BC'.\text{Sin}\,B : CC'$
$\text{Sin}\,CB'B = CB'.\text{Sin}\,C : BB'$	$\text{Sin}\,CAA' = CA'.\text{Sin}\,C : AA'$;

whence it follows, of the six left members,

$$\text{Sin}\,ABB'.\text{Sin}\,BCC'.\text{Sin}\,CAA' = \text{Sin}\,ACC'.\text{Sin}\,CBB'.\text{Sin}\,BAA'.$$

To see this, it is merely necessary to perform the equivalent multiplications of the right members of the equations: and it is a beautiful and easy deduction from [34] and [23 L]. The reasoning and results are unaltered in their truth if you put, for $PA'B'C'$, $P_{,}A_{,}B_{,}C_{,}$. This is the general property of three lines drawn from any point P through the angles of a triangle; and is worth remembering,

[59] P...Ang cŭts a' ĭn dŏtA′, &c.,
$AB'.BC'.CA'$'s $AC'.CB'.BA'$; dŏt áltĕrnā′
pron. ăbbĭccá, ăckĭbbá.
Then for lines put oppo. sines:

i.e. If the line through P and A *cuts* a *in dot*ted A', through P and B cuts b in B', &c.; you *dot* the *alternat*e letters in the assertion $AB'.BC'.CA' = AC'.CB'.BA'$; *then* you may *put for* the 6 lines AB', &c., the *sines* of the 6 angles at ABC *oppo*site to them; and it is still true.

If through any point P *lines be drawn through the angles* ABC *of a triangle to meet the opposite sides in* A′B′C′, *then of the six segments* AB′, B′C, CA′, A′B, BC′, C′A, *the product of the first, third and fifth is equal to that of the second, fourth and sixth, and the like holds of the sines of the six angles which stand over the segments at the vertices of the triangle.*

What is proved above concerning a point and a triangle can be readily established by a similar argument of a point and any polygon of an odd number of sides.

59. It frequently happens that a line is required of which we know only expressly one point, as when we were looking for the equations to 'perc. bic. and biCang,' in Art. (56). Suppose that we were in search of the equation of the line which passes through the given point $(x = e, y = i)$, and through the intersection of two given lines

$$ax + by - c = 0 = v, \text{ and } a_1x + b_1y - c_1 = 0 = u.$$

We can write down what we know about the first point thus: let

$$x - e = \phi(y - i) \text{ be the line required:}$$

for this is one of the innumerable lines passing through (e, i), the equation being true at that point, whatever ϕ may be. What value of ϕ belongs to the sought line we know not; but we know the intersection of the two given lines by [10], and that the sought line contains that point: call it (X, Y). Then we are certain that

$$X - e = \phi(Y - i).$$

As ϕ has the same meaning and value in this and the last equation, we deduce by division of equals by equals, without interfering in any way with the values of x and y,

$$\frac{x-e}{X-e} = \frac{y-i}{Y-i};$$

ϕ is now eliminated, and the equation required is found. Putting for X and Y their values [10], this is

$$\frac{x-e}{\dfrac{c_1b - cb_1}{a_1b - ab_1} - e} = \frac{y-i}{\dfrac{c_1a - ca_1}{b_1a - ba_1} - i}, \text{ or}$$

dividing both sides of the equation by $(a_1b - ab_1)$,

$$\frac{x-c}{(c_1b - cb_1) - e(a_1b - ab_1)} = \frac{y-i}{-c_1a + ca_1 - i(a_1b - ab_1)}. \quad \text{A.}$$

This line we can now draw by the method of Art. (9), even though the point (X, Y) should be ten thousand miles off.

We might have begun the investigation thus: let

$$ax+by-c=\phi(a_1x+b_1y-c_1), \text{ or } v=\phi\,.\,u$$

be the sought line. This passes evidently through the intersection of the two given lines, whatever ϕ may be, being at that point $0=\phi\,.\,0$, which is true. This equation must be true also at (e, i), i. e.

$$ae+bi-c=\phi(a_1e+b_1i-c_1), \text{ must be true,}$$

whence by division comes

$$\frac{ax+by-c}{ae+bi-c}=\frac{a_1x+b_1y-c_1}{a_1e+b_1i-c_1}. \qquad \text{B.}$$

You may prove for yourself that A and B are the same line: you are to multiply both sides of A by the product of their denominators, and both sides of B by the product of theirs, then transposing you will have in both cases the same expression

$$(b_1ia-c_1a-bia_1+ca_1)x+(a_1eb-c_1b-aeb_1+cb_1)y=a_1ec+b_1ic-aec_1-bic_1.$$

Suppose now that we required the equation of the line which passes through the above intersection (X, Y) and is parallel to the given line $y=ex$. We write

$$y-\phi=ex \text{ for our sought line [5];}$$

and as (X, Y) is a point in it, we are certain that

$$Y-\phi=eX, \text{ although } \phi \text{ is unknown.}$$

By subtraction of equals from equals, we get rid of this unknown quantity, and obtain

$$y-Y=e(x-X),$$

the equation required. And you can easily prove from this equation that it gives a line parallel to $y=ex$, and containing the point (X, Y).

Let soughl (pron. saul) stand for *sough*t *l*ine; and say (pron. ϕ, phi.)

[60] Dĭ(xē)'s ϕ. Dĭ(yí) is soúghl thrŏ' ĕí; Di. v. [9].
Nil's vlĕϕú is soúghl throŭgh(vú); ϕu pron. fu.
(y le ϕ)'s ēx, soŭghl párl to y's ēx.

i.e. $x-e=\phi\,.(y-i)$ is a *sough*t *l*ine through the point (e, i); $0=v-\phi u$ is a *sough*t *l*ine through the point (v, u) the intersection of $v=0$ and $u=0$, $y-\phi=ex$ is a *sough*t *l*ine parallel to $y=ex$.

LESSON XVIII.

60. *Jane:*—You have not yet shewn us how to write down the equation of the circle through three given points (x_1y_1), (x_2y_2), and (x_3y_3). I am somewhat curious to see how it will look, and I expect it to be no less entertaining when understood than the equation of a line through two points, so easily acquired in [9].

Uncle Pen.:—You will learn this most agreeably when I have given you a lesson on permutations and combinations; but first we must talk a little about progressions. Do you remember what you have met with on this latter subject in your arithmetic?

Jane:—Indeed I do not; I never understood them.

Richard:—And I never tried, for I do not love the look of them; do give us a trial with some mnemonical aids: I will be very attentive.

Uncle Pen.:—Well then, here is an arithmetical series,

$$a,\ a+d,\ a+2d,\ a+3d,\ldots a+(n-3)d,\ a+(n-2)d,\ a+(n-1)d;$$

the number of its terms is n, and each is made from the next preceding term by the addition of a certain number d, called the *common difference*. The sum of the first and last term is $2a+(n-1).d$, that of the second and last but one is $2a+d+(n-2)d=2a+(n-1).d$, that of the third and last but two is also $=2a+(n-1)d$, and so on. Let us suppose then n is an even number, say $n=2.6=12$: then will there be 6 pairs of terms, first and last, second and last but one, &c. each pair $=2a+(n-1).d$, or if we call the last term z, each pair $=a+z$, the sum of the first and last. The *sum* of all the terms is $6.(a+z)=\frac{n}{2}(a+z)$. Next suppose n odd, say $n=2.6+1=13$. In this case there will be a middle term; call it M. It stands between $M-d$ and $M+d$, which make one of the 6 pairs each equal to $a+z$. M is half the sum of this pair, or $=\frac{1}{2}(a+z)$. The whole series consists of 6 equal pairs $+M$; i. e.

$$6.(a+z)+\tfrac{1}{2}(a+z)=\frac{13}{2}(a+z)=\frac{n}{2}(a+z).$$

Thus whether n be 12 or 13, if S stand for the sum of the series, we have $S = \frac{1}{2} n . (a + z)$. If now you suppose n to be either equal $2m$, any even number, or $2m + 1$, any odd one, and put m for 6 in all the preceding reasoning, you have a demonstration of the theorem,

(a) $$S = \frac{n}{2}(a + z);\ \text{and}$$

(b) $$z = a + (n - 1) . d,$$

is evident of itself, and may be written

$$a - z = d - dn, \text{ by transposition.}$$

[60] If ábc...főrth ĭs Ari.Se:
Ult and a is pěnúlt. and b;
Sum Ari.'s hălf n.Sŭm(áz),
and (d lĕ dn) is Dif.(áz). *dn* a dissyl.

i.e. if $a, b, c, \ldots\ldots$, and so *forth*, be the terms of an *ari*thmetic *se*ries, the (*ult*imate term $+ a$) = the (*penult*imate $+ b$), and so on. The *sum* of the *ari*thmetic *se*ries is always $\frac{1}{2}n$ times the *sum* $(a + z)$, and $(d - dn)$ = the *Diff.* $(a - z.)$

By the equations (a) and (b) we can determine any two of the five quantities S, d, a, z, n, if the other three are given. This gives rise to ten different problems, for the two unknowns may be any of the ten pairs, S and d, S and a, S and z, S and n, d and a, d and z, d and n, a and z, a and n, z and n. If S and n are unknown, and a z and d given, we obtain first n from (b) by transposition and division,

$$n = \frac{d - a + z}{d};\ \text{then putting this for } n \text{ in (a),}$$

$$S = \frac{d - a + z}{2d}(a + z).$$

What is the number of terms, and the sum of the arithmetic progression whose first term is 1, last term 10, and common difference $1\frac{1}{2}$? These formulæ give

$$n = \frac{\frac{3}{2} - 1 + 10}{\frac{3}{2}} = \frac{3 + 18}{3} = 7,\ \text{and}\ S = \tfrac{7}{2}(1 + 10) = 38\tfrac{1}{2}:$$

the series is 1, 2·5, 4, 5·5, 7, 8·5, 10, whose sum is $38\frac{1}{2}$.

What is the number of terms and sum of the series whose first term is 10, last term 1, and common difference $-1\frac{1}{2}$?

The answer is as before $n = 7$, $S = 38\frac{1}{2}$; for neither n nor S is altered in value, if you exchange z and a, and make d negative. The series is the former one read backwards. This is the solution of the fourth of the 10 problems above mentioned: and you will find it a profitable exercise to prove that the solutions of the other nine are in order from the first, as follows:

1.	$2S = n(a+z)$,	$d = (a-z):(1-n)$.
2.	$2S = n.\{2z-d(n-1)\}$,	$a = z-d.(n-1)$.
3.	$2S = n.\{2a+d(n-1)\}$,	$z = a+d.(n-1)$.
5.	$d = (2S-2nz):(n-n^2)$,	$a = (2S-nz):n$.
6.	$d = (2an-2S):(n-n^2)$,	$z = (2S-na):n$.
7.	$d = (a^2-z^2):(a+z-2S)$,	$n = 2S:(a+z)$.
8.	$a = S:n+\frac{1}{2}(d-dn)$, [28]	$z = S:n-\frac{1}{2}(d-dn)$.
9.	$n = [2z+d\pm\{(2z+d)^2-8Sd\}^{\frac{1}{2}}]:2d$,	$a = z+d.(1-n)$.
10.	$n = [d-2a\pm\{(d-2a)^2+8Sd\}^{\frac{1}{2}}]:2d$,	$z = a-d.(1-n)$.

The reason why the ninth and tenth problems require the solution of a quadratic equation is, that (a) with (b) is of the *second degree* in the unknown quantities, containing their product, an and zn. If (a) had happened to contain both n and d, the seventh would have been also a quadratic problem, because the product dn occurs in (b).

In the ninth, n must be of course an integer; therefore the data z d and S must be such that $\{(2z+d)^2-8Sd\}$ shall be a square number M^2, and that $2d$ shall divide without remainder either the sum or the difference of $2z+d$ and M, or else both. When it so divides both, and both quotients are positive, there are two values of n, and consequently two of a, found by putting for n its values successively in the equation $a = z + d(1-n)$.

What is the first term and the number of terms of the series whose sum is 67, its last term 18, and its common difference $2\frac{1}{4}$? Here

$$(2z+d)^2 - 8Sd = (36+2{\cdot}25)^2 - 8\times 67 \times 2{\cdot}25 = 257{\cdot}0625,$$

which is not any square M^2, and there is consequently no number n assignable; that is, no such arithmetic series ex-

ists. But if we put $67\frac{1}{2}$ for S in the question, the data are found to be congruous: we have $(15{\cdot}75)^2 = M^2$, and

$$n = \frac{36+2{\cdot}25 \pm \sqrt{(38{\cdot}25)^2 - 8\times 67{\cdot}5\times 2{\cdot}25}}{2\times 2{\cdot}25} = \frac{54}{4{\cdot}5} \text{ or } \frac{22{\cdot}5}{4{\cdot}5} = 12 \text{ or } 5.$$

Thus there are two values of n; and

$$a = 18 + 2{\cdot}25\times(1-12) = -6\tfrac{3}{4}, \text{ or } a = 18 + 2{\cdot}25\,.(1-5) = 9.$$

There are in fact two series having

$$z = 18,\ d = 2{\cdot}25, \text{ and } S = 67{\cdot}5, \text{ viz.}$$

$9,\ 11\frac{1}{4},\ 13\frac{1}{2},\ 15\frac{3}{4},\ 18$; and $-6\frac{3}{4},\ -4\frac{1}{2},\ -2\frac{1}{4},\ 0,\ 2\frac{1}{4},\ 4\frac{1}{2},\ 6\frac{3}{4},\ 9,\ 11\frac{1}{4},\ 13\frac{1}{2},\ 15\frac{3}{4},\ 18$; the first of 5, the second of 12 terms.

If the three numbers $\frac{1}{a}, \frac{1}{m}, \frac{1}{b}$, are an arithmetical progression, they must be of the form $\frac{1}{m} - d,\ \frac{1}{m},\ \frac{1}{m} + d$, for some value positive or negative of d. The middle term is evidently half the sum of the two extreme terms, i.e.

$$\frac{1}{m} = \tfrac{1}{2}\left(\frac{1}{a} + \frac{1}{b}\right) = \tfrac{1}{2}\cdot\left(\frac{b}{ab} + \frac{a}{ab}\right) = \frac{b+a}{2ab}.$$

You have learned already by definition (42) that $2ab{:}(b+a)$ is the harmonic mean between a and b; but this mean is m; hence if $\frac{1}{a}, \frac{1}{m}, \frac{1}{b}$ form an arithmetic progression, a, m, b, form an harmonic progression. The reciprocals of the arithmetical series taken in the same order form an harmonic series; and *generally, if abcd...to* n *terms, are a progression of the former,* $\frac{1}{a}\ \frac{1}{b}\ \frac{1}{c}\ \frac{1}{d}$... *the reciprocals of the same* n *terms, are said to be a progression of the latter kind.* Thus $\frac{1}{2}$, $2\frac{3}{4}$, 5, are an arithmetic, and 2, $\frac{4}{11}$, $\frac{1}{5}$, are an harmonic progression. You may add to the preceding mnemonic this line

[60'] cips Ari.Se. are Harmo.Se. cips for reciprocals of.

61. A geometrical progression of t terms is seen in the following series:

$$a,\ ae,\ ae^2,\ ae^3 \ldots\ldots ae^{t-3},\ a{\cdot}e^{t-2},\ a{\cdot}e^{t-1},$$

in which e is *the ratio,* and every term after the first is e

times the preceding one, a and e being any positive or negative numbers. If we put z for the last term, and S for the sum of the t terms, we have

$$S = a + ae + ae^2 + ae^3 + \ldots\ldots + ae^{t-3} + ae^{t-2} + z, \text{ and}$$

$$eS = ae + ae^2 + ae^3 + \ldots\ldots + ae^{t-3} + ae^{t-2} + z + ze, \text{ whence}$$

$$S - eS = a - ze, \text{ and } S = (a - ze):(1 - e).$$

We have here, as in the last Article, five quantities, S, a, z, t, e, viz. the sum of the series, its first and last terms, the number of terms and the ratio, and two equations about them:

$$\text{(G)} \qquad z = ae^{t-1}, \quad S = \frac{a - ze}{1 - e}.$$

Ten problems may be proposed, by considering in turn every pair of these five quantities as unknowns, viz. the pairs

$$zS,\ eS,\ aS,\ az;\quad ae,\ ze;\quad ta,\ tz,\ te,\ tS.$$

The first is already solved; the second gives, [47, d],

$$e = (z:a)^{\frac{1}{t-1}},\ S = (z^{\frac{t}{t-1}} - a^{\frac{t}{t-1}}):(z^{\frac{1}{t-1}} - a^{\frac{1}{t-1}}).$$

The third and fourth are easy. The fifth and sixth you may attempt, but you will arrive at an equation beyond your present power to solve. The remaining four are readily managed by the application of [48 b]. Thus, when t and e are the unknowns,

$$e = \frac{S - a}{S - z};\quad z = a\left(\frac{S - a}{S - z}\right)^{t-1},\quad \frac{S - a}{S - z}\,z = a\left(\frac{S - a}{S - z}\right)^{t},$$

whence

$$\log(S - a) - \log(S - z) + \log z = \log a + t.\log(S - a) - t\log(S - z),$$

from which t is readily found after transposition and division.

When e is a positive proper fraction and t is a very great number, e^{t-1} is very small; if $e = \cdot 1$ and $t - 1 = 100000$, $z = ae^{t-1}$ = a millionth part of a, and z diminishes, as t, the number of terms, increases. When t is infinite, $z = 0$, and $ze = 0$; i. e. $S = \dfrac{a}{1 - e} = a + ae + ae^2 + ae^3 + \ldots$ *ad infinitum*, by (G), so that $\dfrac{a}{1 - e}$ represents a geometrical series of an infi-

nite number of terms. Thus, if $a = 1$, $e = 0{\cdot}1$, $t =$ infinite, $\frac{1}{1 - 0{\cdot}1} = \frac{10}{9} = 1 + {\cdot}1 + {\cdot}01 + {\cdot}001 + {\cdot}0001 + \ldots$&c. for ever, as you know already by decimal arithmetic. And, [3], if the lower sign on the left be taken with the lower signs on the right,

$$\text{(H)}\quad \frac{1}{1 \mp e} = 1 \pm e + e^2 \pm e^3 + e^4 \pm e^5 \ldots \textit{ad infin.}\ (e \text{ a prop. frac.}).$$

Here you must remember that e is a proper fraction, positive or negative; when $e = 1$, the series first written becomes $a + a + a + a + \ldots$*ad infin.*, which is in fact no *progression* at all. You are not to consider $\frac{a}{1 - e}$ as representing *numerically* an infinite geometrical progression, for any values of e which do not lie *between* $e = 1$ and $e = -1$, nor for $e = 0$.

Let us call e^{t-1}, for ear-memory's sake, e tos, to power s, (tos one syl.) s being the letter of the alphabet one short of t.

[61] (G) z's á of (ĕ tos); nŭm. térm. is t ;
of = times, v. [38.]
ă lĕ zé by D(ŭné) is Sum Geo. Se.
D = Diff. un = unity.

(H) One bў D(ŭné) ĭs frȣ ŏn é,
is Súm, ĭf ĕ's frác. of an infi. Se.

i.e. $z = a \times$ (e to s power); the *num*ber of *terms* is $(s+1)$ or t; {a *le*ss (*ze*)} by *D*iff. (*un*ity $- e$) = the sum of *geo*m. *se*ries whose first and last terms are a and z, and the ratio, e.

On e, or e *on*, is $e + e^2 + e^3 + \ldots\ldots$ and so *on*, with increasing indices. Frȣ on e, is *e on from* *ουδεν* power or $e^0 + e' + e^2 + \ldots\ldots$ vid. ȣ [53] and (46.) One by Diff. $(1 - e)$ is $e^0 + e' + e^2 + \ldots\ldots$, and is the sum, if e is a proper *frac*tion (pos. or neg.), *of* an *infi*nite *se*ries.

Examples on these progressions may be met with in most treatises on Arithmetic.

LESSON XIX.

62. Of three symbols 1, 2, 3, you can make six = 1.2.3 different permutations and no more, 123, 231, 312, 132, 321, 213. With each of these and a new element 4, you can make four permutations, as with the second you obtain

2314, 2341, 2431, 4231. This gives $1 . 2 . 3 \times 4$ permutations. Again, with each of these and a new element 5 can be found five different permutations of 5 elements, making in all $1 . 2 . 3 . 4 . 5$: and thus it is easily proved by continuing the argument to a new element p, that of p symbols the different permutations are in number

$$1 . 2 . 3 . 4 \ldots (p-2) . (p-1) . p.$$

Suppose p to be 6, you have $1 . 2 . 3 . 4 . 5 . 6$ permutations. Take up any one of them, as 142365, and collect under it those which differ from it only in the mutual arrangement of 1 2 3. You will have 243165, 341265, 143265, 342165, 241365 to write under it: and you may place by the side of this a second row of permutations which differ in nothing but the mutual arrangement of 1 2 and 3; and so on till the $1 . 2 . 3 . 4 . 5 . 6$ permutations are distributed into rows of six, each six all alike save in the mutual position of 1 2 and 3. If now you put 3 for 1, and 3 for 2 also, throughout, your six elements become 333456, and every row of 6 becomes 6 repetitions of one permutation. The number of different arrangements is now only equal to that of the rows, or $\frac{1}{6}$ of what it was at first; it is $\dfrac{1 . 2 . 3 . 4 . 5 . 6}{1 . 2 . 3}$, and this is the number of different permutations which can be made with the six elements 333456. If you now take up any one of these 120 permutations, as 343365, you will find another to place under it, 353364, which differs from it only in the mutual position of 4 and 5, and thus you will have 60 pairs each alike, save as to 4 and 5. If now you put 5 for 4 throughout, your six elements are 333556, and the number of different permutations is half what it was: it is exactly $\dfrac{1 . 2 . 3 . 4 . 5 . 6}{1 . 2 . 1 . 2 . 3}$.

By repeating this argument for any value of p, you prove:

If p *elements contain* m a's, e b's, i c's, *the different permutations of those* p *elements are in number*

$$\frac{1 . 2 . 3 \ldots\ldots (p-2) . (p-1) . p}{1 . 2 . 3 \ldots (m-1) . m . 1 . 2 . 3 \ldots (e-1) . e . 1 . 2 . 3 \ldots (i-1) . i}.$$

The permutations of *aaaaabbbbcccdddef* amount to

$$\frac{1.2.3.4.5.6.7.8.9.10.11.12.13.14.15.16.17}{1 . 2 . 3 . 4 . 5 . 1 . 2 . 3 . 4 . 1 . 2 . 3 . 1 . 2 . 3}.$$

Let *fags* stand for *fa*ctor di*g*it*s*: then $1 . 2 . 3$ is 3 *fags*,

1 . 2 . 3 . 4 . 5 is 5 *fags*, &c.: and you may say the above proposition thus:

[62] If p ĕléms. hăve m ă's, é b's, i c's,
The perms. in p's are p fags bў (m fags. e fags. i fags).

63. There are many *permutations* of the same *combination*. Thus *abcd*, *acdb* are the same *combination*, but different permutations of it. Suppose six symbols, 123456; with each of them you can combine in turn every other, as

12, 13, 14, 15, 16; 23, 24, 25, 26, 21; 34, 35, 36, 31, 32;

and thus can be completed six fives; but every pair is twice written, as 12 and 21, so that the exact number of combinations two together that can be made out of 6 elements is $\frac{1}{2} . 6 . 5$. In the same way you can make with n elements $\frac{n.(n-1)}{2}$ duads, or combinations of two. Again, each of the 15 duads made with 6 symbols can be combined with the remaining 4, as from 12 are made the four, 123, 124, 125, 126, and thus you can complete 15 such fours; but 123 will be thrice written, being made once from 12, once from 23, and once from 31. The correct number of combinations of threes made out of 6 is one third of those 15 fours, or $\frac{6 . 5 . 4}{1 . 2 . 3}$. In the same way can be proved that with n elements can be made triplets in number

$$\frac{n . (n-1) . (n-2)}{1 . 2 . 3}.$$

With every different triplet can be combined each of the remaining $n-3$ symbols, thus forming

$$\frac{n . (n-1) . (n-2)}{1 . 2 . 3} \times (n-3)$$

quadruplets or combinations of four; but each of these will be made four times over, each time by adding a different fourth single symbol; and this consideration reduces the number of different 4-plets to

$$\frac{n . (n-1) . (n-2) . (n-3)}{1 . 2 . 3 . 4},$$

the fourth part of the preceding. By carrying on this mode of reasoning, it is easy to prove that

The number of non-repeating combinations of d *together, that can be formed with* n *symbols, is*

$$\frac{n.(n-1).(n-2)...\{n-(d-1)\}}{1.2.3...(d-1).d}.$$

We may call the quantity $9.8.7$, three nine-backs, and $9.8.7.6.5$, five nine-backs; $n.(n-1).(n-2)$ is three n-backs, and the numerator just written is d n-backs. By non-repeating combinations are meant such as contain no repeated letter: *abcde* is a non-repeating, and *aabcd* is a repeating combination. You may add to your stock of mnemonics the following: (com. for combinations, repea. for repeating).

[63] Comb. nón-repéa.'s of n in d's,
Are d n-bácks by d fags. fags vid. [62].

The number of repeating combinations of n *things taken* d *together, is*

(the proof follows below)

$$\frac{n.(n+1).(n+2)...\{n+(d-1)\}}{1.2.3...(d-1).d}.$$

And you may join this mnemonic to the preceding: (repe. combs. for repeating combinations).

[64] The répe. combs. of n in d's,
Are d n-úps by d fags.

Three 5-ups is $5 \cdot 6 \cdot 7$; 4 n-ups $= n \cdot (n+1) \cdot (n+2) \cdot (n+3)$.

Among the repeating combinations of 7 in 4's are counted 2222, 2776, 4441, &c., as well as all the non-repeating combinations of 7 in 4's. The proof of this proposition [64] is not so easy as that of the preceding, and you will perhaps have to meditate it somewhat longer than those. Now, I say, that the theorem is true for any values N and D, of n and d, if it be true for all values of n and d both less than D. We are seeking the number of repeating D-plets that can be made with N symbols: and our hypothesis shall be that for values of n and d both less than D, [64] is true.

64. Let D be divided into any two positive numbers, m and e, so that m not $> N$, and *suppose* all the non-repeat-

ing D-plets to be written out that are possible with $(N+D-1)$ things. Their number, by [63], is

$$D\ (N+D-1)\text{-backs by } D \text{ fags, or}$$

$$\frac{(N+D-1)(N+D-2)\ldots\{N+D-(D-1)\}(N+D-D)}{1.2.3\ldots(D-1).D},$$

which is D N-ups by D fags also, the same fraction with that over [64], when $n=N$ and $d=D$. We may suppose our $N+D-1$ symbols to be letters in alphabetical order. Let A represent any D-plet of those just written out, which contains m letters and no more of the first N: A will contain e of the remaining $D-1$ letters, because $D=m+e$; and the same m letters of A will form part of D-plets A, A_1, A_2, A_3, &c. in number equal to the non-repeating e-plets that can be made of those $D-1$ remaining letters. That is, this m-plet, which we may call a, will appear

$$\frac{(D-1)(D-2)(D-3)\ldots\{D-1-(e-2)\}\{D-1-(e-1)\}}{1.2.3\ldots(e-2).(e-1).e}$$

times, by [63], or, since

$$m=D-e,\quad m+e-1=D-1,$$

$$\frac{m.(m+1).(m+2)\ldots\{m+(e-2)\}\{m+(e-1)\}}{1.2.3\ldots(e-1).e}\ \text{times,}$$

which is e m-ups by e fags, the exact number of repeating combinations of m symbols e together, as, by hypothesis, we know beforehand; m and e being both less than D, whatever numbers they may be. If we now erase all the letters, that lie alphabetically beyond the first N, from these D-plets, A, A_1, A_2 &c., we can replace the erased e-plets by the repeating combinations e together of the m letters in a, and we shall have in place of A, A_1, &c., all the D-plets that can be made by adding to a e repeated letters out of its m symbols. When D is greater than N, our restriction, m not $>N$, makes it necessary that e not $<(D-N)$. As e will have every value less than D, and not $<(D-N)$, we shall, after thus collecting together all the D-plets of those above supposed written that contain m of the first N letters, for all values of m not $>N$ and $<D$, and substituting repeated letters out of those m for the letters beyond the first N, we shall, I say, thus have before us every D-plet possible with N things, in which there is any $D-1$ repeated letters, or any $D-2$ repeated

letters, or any e repeated letters, whatever e may be. We shall have also, among the system supposed written out, all the non-repeating D-plets *possible* out of N letters. But we have made no change in the number of the D-plets by these erasures and substitutions, which is still

$$\frac{N.(N+1).(N+2)\ldots\{N+(D-2)\}(N+D-1)}{1.2.3\ldots(D-1).D};$$

and this proves the truth of [64], for $n=N$, $d=D$, if only it be known true for $n=m$ and $d=e$ both less than D. Now we see easily that when $n=2$ and $d=2$, [64] is true; for 2 2-ups by 2 fags $=2\cdot3:1\cdot2=3$, and the repeating combinations of 2 in twos are *aa, ab, bb,* exactly three. Therefore [64] is true, by the preceding argument, for $n=N$, $d=3$; because we do know it to be true for $n=2$ and $d=2$, which are less than 3: consequently it is true for $n=N$, and $d=4$; for we can prove it true for all values of n and d less than 4: and thus proceeding, we can establish its truth for every value of d. In all this N may be any number we choose.

The above argument will be at first perplexing. I advise you to read it aloud putting small numbers for N and D; say $N=6$, $D=3$, and again $N=6$, $D=4$, all through. When D is 3, m and e must be either 1 and 2, or 2 and 1; when D is 4, m and e may be 1 and 3, 3 and 1, or 2 and 2; and when $m=1$, a is of course a single letter.

The repeating combinations of 5 in threes are, in addition to the 10 non-repeating, *aaa aab aac aad aae abb acc add aee bbc bbd bbe bcc bdd bee ccd cce cdd cee dde dee bbb ccc ddd eee*, 25 more, making in all $5\cdot6\cdot7:1\cdot2\cdot3$. Those of 5 in sixes are 210 in number, *aaaaaa, aaaaab, aaaaac, aaaaad, aaaaae, aaaabb*, &c.

65. If you take all the non-repeating combinations of six in threes, and write down the six permutations of each one, you have what are called the non-repeating *variations* of six in threes, which are in number $6\cdot5\cdot4$, viz. that of the combinations [63] multiplied by the number of the permutations of each one, [62].

The non-repeating variations of n *symbols taken* d *together, are* [$\mathrm{n}.(\mathrm{n}-1).(\mathrm{n}-2)\ldots\{\mathrm{n}-(\mathrm{d}-1)\}$].

[65] Non-repe. vars. of n in d's
Are d n-backs.

If to the non-repeating variations of six in twos you add the six repetitions, 11, 22, 33, 44, 55, 66, you obtain *the repeating variations* of six in twos: the number of which is by [65] $6.5+6=6.(6-1)+6=6^2$. And the repeating variations of n in twos are in the same way $n.(n-1)+n=n^2$. By adding to each of these n^2 duads in turn the n symbols, $n.n^2$ triplets are formed, which are the repeating variations of n in threes. Every symbol of the n can be added to each of these, giving $n.n^3=n^4$ 4-plets, the repeating variations of n in fours; and so on.

The repeating variations of n *in* p*'s are in number* n^p.

[66] Thė répe. várs. of n in p's
Are n to pth. pth power

How many different whole numbers of seven figures can be made with the digits 1, 2 and 3? The answer is the number of repeating variations of three in sevens, which is $=3^7=2187$.

LESSON XX.

66. THE product $(1+r_1)(1+r_2)$

$$=1+(r_1+r_2)+r_1r_2=A\ ;\ (1+r_1).(1+r_2).(1+r_3)=(1+r_3).A$$
$$=1+(r_1+r_2+r_3)+(r_1r_3+r_2r_3+r_1r_2)+r_1r_2r_3=B\ ;\quad (1+r_4).B$$
$$=1+(r_1+r_2+r_3+r_4)+(r_1r_4+r_2r_4+r_3r_4+r_1r_3+r_2r_3+r_1r_2)$$
$$+(r_1r_3r_4+r_2r_3r_4+r_1r_2r_4+r_1r_2r_3)+r_1r_2r_3r_4=C.$$

The first bracketed term in C contains the non-repeating combinations of four in ones, the next those of four in twos, the next those of four in threes, the last the only one of four in fours, as is evident from the subindices. If now the product $(1+r_5)\,C$ be formed, r_5 will be combined with the six duads $r_1\,r_4$ &c., making six new triplets $r_1r_4r_5$ &c., in the product, and the four triplets $r_1r_3r_4$ &c. being multiplied by unity, will be added to those six, forming ten or 5.4.3:1.2.3 triplets, which are by [63] all the non-repeating combinations of five in threes. In the same way

it can be shown that the product $(1 + r_5) C$ will contain $5·4·3·2:1·2·3·4$ quadruplets $r_1 r_3 r_4 r_5$ &c., i. e. all the non-repeating combinations of five in fours.

$$\text{And as } \frac{m.(m-1)\ldots\{m-(e-2)\}\{m-(e-1)\}}{1.2\ldots(e-1).e}$$
$$+\frac{m.(m-1)\ldots\{m-(e-2)\}}{1.2\ldots(e-1)}$$
$$=\frac{(m-e+1)[m.(m-1)\ldots\{m-(e-2)\}]}{1.2\ldots(e-1).e}$$
$$+\frac{e[m.(m-1)\ldots\{m-(e-2)\}]}{1.2\ldots(e-1).e}$$
$$=\frac{(m+1).m.(m-1)\ldots\{m-(e-2)\}}{1.2\ldots(e-1).e}$$
$$= e\,(m+1) \text{ -backs by } e \text{ fags,}$$

it is evident by [63] that *the number of non-repeating combinations of* m *in* e's + *that of those of* m *in* (e − 1)'s = *that of those of* (m + 1) *in* e's.

We had $m = 4$ above, and again,

$$\frac{9.8.7.6}{1.2.3.4}+\frac{9.8.7}{1.2.3}=\frac{10.9.8.7}{1.2.3.4}.$$

By forming thus the product $(1 + r_5) C = D$, $(1 + r_6) D = E$, &c., until $(1 + r_n)$ is introduced, it can be proved that

$$(1+r_1)(1+r_2)(1+r_3)\ldots(1+r_{n-1})(1+r_n)$$
$$=1+P_1+P_2+P_3+\ldots+P_i+P_{i+1}+\ldots+P_{n-1}+P_n\,;$$

in which P_i is the sum of the non-repeating combinations of the n letters r_1 r_2 &c., in i's, whatever i may be from $i = 1$ to $i = n$.

All this is true whatever be the numbers r_1 r_2 &c.; let us then suppose them all equal, $r_1 = r_2 = \ldots = r_n$; the sub-indices may be erased now, and the last written equation will be

$$(1+r)^n = 1 + nr + \frac{n.(n-1)}{1.2}r^2 + \frac{n.(n-1).(n-2)}{1.2.3}r^3 + \ldots$$
$$+\frac{n.(n-1).(n-2)\ldots\{n-(i-1)\}}{1.2.3\ldots i}r^i$$
$$+\frac{n.(n-1).(n-2)\ldots(n-i)}{1.2.3.4\ldots i.(i+1)}r^{i+1}+\ldots$$

$$+\frac{n.(n-1).(n-2)\ldots\{n-(n-2)\}}{1.2.3\ldots(n-1)}r^{n-1}$$
$$+\frac{n.(n-1).(n-2)\ldots\{n-(n-1)\}}{1.2.3\ldots n}r^{n};$$

for every pair rr will be now r^2, and the number of these pairs is $\frac{n.(n-1)}{1.2}$ by [63]; every triplet rrr will be r^3, of which there are $\frac{n.(n-1).(n-2)}{1.2.3}$, &c.

Thus, if for n we put in succession the values $n=1$, $n=2$, $n=3$, $n=4$, $n=5$, we obtain from the last equation

$$(1+r)^1 = 1 + r\,;\ (1+r)^2 = 1 + 2r + \frac{2\,.\,1}{1\,.\,2}r^2 \text{ as in [14]},$$
$$(1+r)^3 = 1 + 3r + \frac{3\,.\,2}{1\,.\,2}r^2 + \frac{3\,.\,2\,.\,1}{1\,.\,2\,.\,3}r^3 = 1 + 3r + 3r^2 + r^3,$$
$$(1+r^4) = 1 + 4r + \frac{4\,.\,3}{1\,.\,2}r^2 + \frac{4\,.\,3\,.\,2}{1\,.\,2\,.\,3}r^3 + \frac{4\,.\,3\,.\,2\,.\,1}{1\,.\,2\,.\,3\,.\,4}r^4$$
$$= 1 + 4r + 6r^2 + 4r^3 + r^4,$$
$$(1+r)^5 = 1 + 5r + 10r^2 + 10r^3 + 5r^4 + r^5.$$

If $m=2m+1$, any odd number, there will be an even number of terms, namely, $2m+2$, and no middle term; if $n=2m$, there will be an odd number of terms, namely, $2m+1$, and therefore a middle term; in the former case every coefficient will occur twice, as in the last equation 1, 5, and 10, each occur twice, for the first term, unity, or $1r^0$, may be called the coefficient of the zero power of r; in the latter case every coefficient will occur twice, except that of the middle term in which r^m appears. The reason of this is, that *one* n-*backs by one fag* = n = (n − 1) n-*backs by* (n − 1) *fags; two* n-*backs by two fags* = (n − 2) n-*backs by* (n − 2) *fags*, as you easily convince yourself.

We have now proved the celebrated *Binomial Theorem* of Newton, at least for whole and positive values of the index n; and this theorem is expressed thus, A.

$$(1+r)^n = 1 + nr + \frac{n.(n-1)}{1.2}r^2 + \frac{n.(n-1).(n-2)}{1.2.3}r^3 + \&c.,$$

the &c., denoting that the terms are supposed to be con-

tinued to the right for ever, the powers of r constantly rising by one, and the i^{th} power of r, r^i, being always multiplied by (i n-backs by i fags).

Jane:—Are we to conceive the terms carried on for ever, when $n=1$ or 2 or 3 or 4 or 5? How can this be true when we see, in the examples you have just written, that they do not so go on?

Uncle Pen.:—You are to distinguish between the algebraic form and the arithmetical value of the series. The form is the same for innumerable values of n; for the truth of the equation depends no more on any particular value of n than on the value of r. The number of terms *must be* always the same if the form is the same. Now we can prove that however great an integer n may be, the number of terms is at least $n+1$; i.e. number of terms is greater than any number, however great; it is therefore, no finite number, but infinite. It happens that, when n is a positive whole number $=n'$, the $(n+2)^{\text{th}}$ term and all the succeeding ones become zeros, by reason of the factor $(n-n')$ in the numerator; when $n=1$ the third term $=0$ by reason of $(n-1)$; when $n=2$, the fourth becomes zero by reason of $(n-2)$, &c. But if you put for n on the left side of the equation, a value which is *not* a positive whole number, and the same value for n on the right, none of the terms can vanish, because $(n-i)$, whatever positive integer i may be, cannot be zero for such a value of n. Thus put $n=\frac{3}{4}$;

$$(1+r)^{\cdot 75}=1+\cdot 75r+\frac{\cdot 75(\cdot 75-1)}{1.2}r^2+\frac{\cdot 75(\cdot 75-1)(\cdot 75-2)}{1.2.3}r^3+\&c.$$

is what Newton's theorem becomes, and no term will ever become zero. The question now arises, is this infinite series on the right really equal to $(1+r)^{\cdot 75}$? We have *proved* the truth of this equation only for the case of n whole and positive. It is *just possible*, that there may be, in the expanded value of $(1+r)^{\frac{3}{4}}$, besides the infinite series of the form above written, a certain term T which has a value for $n=\frac{3}{4}$, and which vanishes for every whole and positive value of n. If so, the series on the right will be either greater or less than $(1+r)^{\frac{3}{4}}$, and its real value will be some unknown number R, different from $(1+r)^{\frac{3}{4}}$. For the present you must take it, on my assurance, that this theorem of Newton is perfectly general, and true for all values of r, whole or fractional, positive or negative, possible or impossible. I shall

content myself with giving you a clear notion of the *shape* of the successive terms when n is negative or fractional. To *prove a negative*, e.g. the non-existence of this supposed term T, is a very difficult task; and I strongly suspect that some of the proofs which I have seen in writers on Algebra of this point, are no proofs at all. When n is the fraction $\left(\frac{\mathrm{n}}{\mathrm{e}}\right)$, the fifth term of the series is

$$\frac{n.(n-1).(n-2).(n-3)}{1.2.3.4}r^4=\frac{(\mathrm{n:e})(\mathrm{n:e}-1)(\mathrm{n:e}-2)(\mathrm{n:e}-3)}{1.2.3.4}r^4$$

$$=\frac{(\mathrm{n:e})(\mathrm{n:e}-1)(\mathrm{n:e}-2)(\mathrm{n}-3\mathrm{e})}{1.2.3.4.\mathrm{e}}r^4=\frac{(\mathrm{n:e})(\mathrm{n:e}-1)(\mathrm{n}-2\mathrm{e})(\mathrm{n}-3\mathrm{e})}{1.2.3.4.\mathrm{e}^2}r^4$$

$$=\frac{\mathrm{n}.(\mathrm{n}-\mathrm{e})(\mathrm{n}-2\mathrm{e})(\mathrm{n}-3\mathrm{e})}{1.2.3.4}\cdot\frac{r^4}{\mathrm{e}^4};$$

because the numerator and denominator are both multiplied by the same quantity e.e.e.e, which makes no change in the value of the term. A similar transformation of every term being made, we have

$$\text{B.}\quad (1+r)^{\frac{\mathrm{n}}{\mathrm{e}}}=1+\mathrm{n}\frac{r}{\mathrm{e}}+\frac{\mathrm{n}.(\mathrm{n}-\mathrm{e})}{1.2}\cdot\frac{r^2}{\mathrm{e}^2}+\frac{\mathrm{n}.(\mathrm{n}-\mathrm{e}).(\mathrm{n}-2\mathrm{e})}{1.2.3}\cdot\frac{r^3}{\mathrm{e}^3}$$

$$+\frac{\mathrm{n}.(\mathrm{n}-\mathrm{e}).(\mathrm{n}-2\mathrm{e}).(\mathrm{n}-3\mathrm{e})}{1.2.3.4}\cdot\frac{r^4}{\mathrm{e}^4}+\ldots$$

which differs from the expansion of $(1+r)^{\mathrm{n}}$ in these two points—first, that it proceeds by the powers of $\frac{r}{\mathrm{e}}$ instead of those of r, and secondly, that every digit 1 2 3 4 &c. in the *numerator* of any term, is multiplied by e. When e is negative, the minus signs in all the numerators will of course become plus signs, for $-(-\mathrm{e})$ is $+\mathrm{e}$; and if r be not also negative, every *odd* term, first, third, &c. will be negative in (B), because $r{:}(-\mathrm{e})=-r{:}\mathrm{e}$.

If both r and e are negative while n is positive, every term of the series (B) will be positive, and the signs in the numerators will all be positive. But whatever be the signs of r n and e, you can always obtain the correct expansion of $(1+r)^{\mathrm{n:e}}$, if you put for those three symbols their proper signs and values in the last written equation. To develope or expand $(1+r)^{a}$, you first write out

$$r^0+\quad r^1+\quad r^2+\quad r^3+\quad r^4+\ldots\&\text{c.}\quad (r^0=1),$$

then multiply r^i by (i n-backs by i fags). If you have to develope $(1+r)^{\frac{n}{e}}$, you first write out

$$\frac{r^0}{e^0}+\ \frac{r^1}{e^1}+\ \frac{r^2}{e^2}+\ \frac{r^3}{e^3}+\ \frac{r^4}{e^4}+\&c.,$$

and then multiply r^i by an expression differing from (i n-backs by i fags) in this, that the subtracted digits, 1 2 3 &c. in the numerator, are multiplied by e. To remember (A) and (B), say

[67]	Sŭtón(ŭn r)? wrĭte frȣ ŏn r;	v. [61 H.] frȣ *on.*
	Theŭ r to í you multiply	(A).
	By (i n-bácks bў i fags):	vid. [62], [63].
	If n hăs dĕn.é,	(B).
	Put r vĭ(ré); top dits wĕd e.	vi vid. [6].

Do you want the expansion of *Sum to* power *n* of *un*ity and *r* ? Write &c. : *r* to *i* is r^i (to power *i*). There is no fear of your mistaking this for the ratio $r{:}i$. *Den*. for *den*ominator. If the index has *e* for its *den.*, you write frȣ on vi(*re*) for frȣ on *r*. vi(*re*) $=r{:}e$; and the *top digits* (of the *numerators*) 1, 2, 3, &c., *wed e*, i.e. are multiplied by *e*, as in (B).

67. Examples of expansions when n is fractional are,

$(1\pm r)^{\frac{1}{-1}}=1\mp r+r^2\mp r^3+r^4\mp\&c.$, as in [61 H] v. [44]

$$(1-r)^{\frac{3}{-1}}=(1-r)^{-3}=1+3r+6r^2+10r^3+15r^4+\ldots$$

$$(1-r)^{\frac{1}{3}}=1-\frac{1}{3}r-\frac{1}{9}r^2-\frac{5}{81}r^3-\&c.\ldots$$

All these you may prove from (B) by making the requisite substitutions for n and e, and then simplifying.

You will now find no difficulty, except the mere length of arithmetical operations, in expanding $(a+b)^{\frac{n}{e}}$, whatever numbers a b n and e may be. This is called a *binomial* quantity; $(a+b+c)^m$ is the m^{th} power of the *trinomial* $(a+b+c)$; a binomial or a trinomial is a quantity consisting of two or of three terms. Since

$a(1+b{:}a)=(a+b)$, $(a+b)^m=a^m\{1+b{:}a\}^m$, by [47, c],

you have only to put $(b{:}a)$ for r in (A), to expand $(1+b{:}a)^m$, and then to multiply every term by a^m. For example,

$$(a+b)^2=a^2(1+b{:}a)^2=a^2(1+2b{:}a+b^2{:}a^2)=a^2+2ba+b^2,$$

$$(a+b)^3 = a^3\left(1+\frac{b}{a}\right)^3 = a^3\left(1+3\frac{b}{a}+3\frac{b^2}{a^2}+\frac{b^3}{a^3}\right)$$
$$= a^3+3a^2b+3ab^2+b^3,$$

$$(a+b)^4 = a^4\left(1+\frac{b}{a}\right)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4,$$

$$(a+b)^5 = a^5\left(1+\frac{b}{a}\right)^5 = a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5,$$

$$(a-b)^{\frac{1}{3}} = a^{\frac{1}{3}}\left(1-\frac{b}{a}\right)^{\frac{1}{3}} = a^{\frac{1}{3}} - \frac{1}{3}b\,.\,a^{-\frac{2}{3}} - \frac{1}{9}\,.\,b^2a^{-\frac{5}{3}} - \frac{5}{81}\,b^3a^{-\frac{8}{3}} - \&\text{c.}$$

$$(a \pm b)^n = a^n\left\{1 \pm \frac{b}{a}\right\}^n = a^n \pm na^{n-1}b + \frac{n\,.\,(n-1)}{1\,.\,2}\,a^{n-2}b^2$$
$$\pm \frac{n\,.\,(n-1)\,.\,(n-2)}{1\,.\,2\,.\,3}\,a^{n-3}b^3 + \&\text{c.}$$

In all these you observe that the sum of the indices of a and b in any term is equal to the index of the binomial; as in the last line but one $1-\frac{2}{3}=\frac{1}{3}$, $2-\frac{5}{3}=\frac{1}{3}$, &c., and in the last line, $n+0=n$, $n-1+1=n$, $n-2+2=n$, &c. You may exercise yourself to prove the following expansion of the cube root of 31;

$$(31)^{\frac{1}{3}} = (27+4)^{\frac{1}{3}} = 27^{\frac{1}{3}}\left\{1+\frac{4}{27}\right\}^{\frac{1}{3}} = 3\,\{1+\frac{4}{81}-\frac{16}{6561}$$
$$+\frac{320}{1594323}-\frac{2560}{129140163}+\ldots\} = 3+0{\cdot}14815-0{\cdot}00731$$
$$+0{\cdot}0006-0{\cdot}00006+\ldots = 3{\cdot}14138\,;$$

a result correct to the last decimal; and this correctness might be increased by taking a greater number of terms of the series. In this expansion the terms diminish rapidly in value, and therefore a small number of them will give a tolerably accurate result, the rest of the infinite series being so small that it may be neglected. Such a series is said to be *converging;* but not every infinite series is of this kind: many are *diverging,* and of no use for arithmetical calculation. On this subject there are very many curious and interesting things known to mathematicians, and many more yet to be discovered.

LESSON XXI.

68. We have now cleared our way fairly into the most attractive fields of geometry, and are about to commence in earnest the study of the circle and those other curves of the same family, whose beautiful properties were for so many centuries the delight of the Indian, the Egyptian, and the Grecian sages of the olden time. You were curious, my dear Jane, to see how the equation to the circle through three points (x_1y_1) (x_2y_2) (x_3y_3) would look, and to compare it with that to the line through two given points. This last has a secret to reveal to us of immense importance, if we arrange its terms in the fashion following:

$$x_0y_1 1_2 - x_0y_2 1_1 + x_1y_2 1_0 - x_1y_0 1_2 + x_2y_0 1_1 - x_2y_1 1_0 = 0, \qquad (a)$$

which is merely the equation (8) of Art. (16), the units being introduced for symmetry only. Looking at the subindices, you see every permutation of 012, and observe that every two terms, whose subindices differ by the exchange of a single pair, are of opposite signs; as the first and second, differing by the exchange of 1 and 2, the first and last, differing by that of 0 and 2, as also the second and fifth. The effect of this arrangement is, that if you suppose any two subindices to coincide in value, the whole expression is reduced to pairs of terms that destroy each other. Suppose 0 and 1 to coincide, in other words, let $x_0 = x_1$ and $y_0 = y_1$; then the first and fourth terms destroy each other, as do the second and third, and the fifth and sixth; which proves that the line represented passes through the point $(x = x_1, \ y = y_1)$. The same thing happens if 0 and 2, or if 1 and 2 be supposed to coincide in value. This last supposition is of course only admissible by defining (x_0y_0) as one of the two given points, and either (x_1y_1) or (x_2y_2) to be the co-ordinates of the current variable point.

Further, no two terms which have the same sign can be made alike by the exchange of a single pair of subindices. Thus $x_0y_1 1_2$ requires two exchanges, 102 and 120, before it can be made the same with the third term, first that of 0 and 1, then that of 0 and 2. We see then that in this equation the whole of the $1 . 2 . 3$ permutations of 012 occur [62], and that the signs are determined by this law, that

two terms which can be made to agree in their subindices by one exchange of one pair, performed in one term, have opposite signs; while two terms, which cannot be made so to agree without performing two such exchanges, have the same sign.

If we put x^2 for every x the expression becomes

$$x_0^2 y_1 1_2 - x_0^2 y_2 1_1 + x_1^2 y_2 1_0 - x_1^2 y_0 1_2 + x_2^2 y_0 1_1 - x_2^2 y_1 1_0 = 0, \quad \text{(b)}$$

which vanishes like the preceding if

$$x_0 = x_1 \text{ and } y_0 = y_1, \text{ or if } x_0 = x_2 \text{ and } y_0 = y_2,$$

and for the same reason, being then reduced to three pairs of terms, each pair equal zero. But this is no equation to a right line, for it is of the form

$$Ax_0^2 + By_0 + C = 0, \text{ or } x^2 = (By + C):A,$$

where ABC are constant numbers; and gives

$$x = \pm\sqrt{(By + C):A},$$

yielding two values of x for every value of y. Thus if $y = 0$, which is true only in the axis of x, (11),

$$x = \pm\sqrt{C:A},$$

so that this locus has two points in the axis of x, equidistant from the origin. It is, however, not a circle, (28), because the coefficients of x^2 and y^2 are not alike. It is in fact,

$$A^2x^2 + 0 \cdot y^2 + By + C = 0,$$

zero being the coefficient of y^2. Look now at the expression

$$(y_0^2 + x_0^2)\, y_1 1_2 - (y_0^2 + x_0^2)\, y_2 1_1 + (y_1^2 + x_1^2)\, y_2 1_0 - (y_1^2 + x_1^2)\, y_0 1_2$$
$$+ (y_2^2 + x_2^2)\, y_0 1_1 - (y_2^2 + x_2^2)\, y_1 1_0 = 0: \quad \text{(c)}$$

the arrangement of subindices is exactly as before. If you suppose $x_0 = x_1$ and $y_0 = y_1$, the first and fourth term, the second and third, the fifth and sixth are self-destroying pairs, as in (a) and (b): wherefore $(x_1 y_1)$ is a point of this locus, the equation being *satisfied* at that point; and if we put $x_0 = x_2$, $y_0 = y_2$, we see in the same way that $(x_2 y_2)$ is also a point of it. This equation is exactly

$$(y_1 - y_2)\, y^2 + (y_1 - y_2)\, x^2 + (y_2^2 + x_2^2 - y_1^2 - x_1^2)\, y$$
$$+ (y_1^2 + x_1^2)\, y_2 - (y_2^2 + x_2^2)\, y_1 = 0, \quad \text{(c')}$$

of the form H (29), and is therefore, when the axes are rectangular, the equation of a circle (28) which passes through $(x_1 y_1)$ and $(x_2 y_2)$, and has its centre in the axis of x, with radius

$$=\tfrac{1}{2}\left\{\frac{y_2^{\,2}+x_2^{\,2}-y_1^{\,2}-x_1^{\,2}}{(y_1-y_2)^2}-4\,\frac{(y_1^{\,2}+x_1^{\,2})\,y_2-(y_2^{\,2}+x_2^{\,2})\,y_1}{y_1-y_2}\right\}^{\frac{1}{2}}.$$

This becomes

$$\frac{0}{0} \text{ if } y_1=y_2 \text{ and } x_1^{\,2}=x_2^{\,2},$$

an indeterminate quantity, having any one value as much as any other. This shews that if $(x_1 y_1)$ be (3, 4), (e.g.) and $(x_2 y_2)$ be (−3, 4), an infinite number of circles can be drawn to have their centres in the axis of x, and to pass through the two points: and it is obvious, if you draw the figure, that every point in that axis is equidistant from them.

We have seen that the equation of the line through any two points $(x_1 y_1)$ $(x_2 y_2)$ exhibits all the permutations of 012, with a certain law of signs: this makes it *probable* that the equation of the circle through any three points $(x_1 y_1)$ $(x_2 y_2)$ $(x_2 y_2)$ will exhibit all the permutations of 0123, arranged by the same law of signs. At least it is worth while to examine this.

69. Let the co-ordinates be rectangular: we know that in the circle y^2 and x^2 must have the same multiplier. Let us form our equation thus, $0 =$

$$\begin{aligned}
&+(y_0^{\,2}+x_0^{\,2}).y_1.x_2.1_3-(y_0^{\,2}+x_0^{\,2}).y_1.x_3.1_2+(y_0^{\,2}+x_0^{\,2}).y_2.x_3.1_1\\
&\qquad-(y_0^{\,2}+x_0^{\,2}).y_2.x_1.1_3+(y_0^{\,2}+x_0^{\,2}).y_3\,x_1.1_2-(y_0^{\,2}+x_0^{\,2}).y_3.x_2.1_1\\
&-(y_1^{\,2}+x_1^{\,2}).y_2.x_3.1_0+(y_1^{\,2}+x_1^{\,2}).y_2.x_0.1_3-(y_1^{\,2}+x_1^{\,2}).y_3.x_0.1_2\\
&\qquad+(y_1^{\,2}+x_1^{\,2}).y_3.x_2.1_0-(y_1^{\,2}+x_1^{\,2}).y_0.x_2.1_3+(y_1^{\,2}+x_1^{\,2}).y_0.x_3.1_2\\
&+(y_2^{\,2}+x_2^{\,2}).y_3.x_0.1_1-(y_2^{\,2}+x^2_{\,2}).y_3.x_1.1_0+(y_2^{\,2}+x_2^{\,2}).y_0.x_1.1_3\\
&\qquad-(y_2^{\,2}+x_2^{\,2}).y_0.x_3.1_1+(y_2^{\,2}+x_2^{\,2}).y_1.x_3.1_0-(y_2^{\,2}+x_2^{\,2}).y_1.x_0.1_3\\
&-(y_3^{\,2}+x_3^{\,2}).y_0.x_1.1_2+(y_3^{\,2}+x_3^{\,2}).y_0.x_2.1_1-(y_3^{\,2}+x_3^{\,2}).y_1.x_2.1_0\\
&\quad+(y_3^{\,2}+x_3^{\,2}).y_1.x_0.1_2-(y_3^{\,2}+x_3^{\,2}).y_2.x_0.1_1+(y_3^{\,2}+x_3^{\,2}).y_2.x_1.1_0. \quad \text{(d).}
\end{aligned}$$

70. The law of the signs in (a) may be thus stated: *the permutations made by going round any triplet*, or as it is said by *cyclically permuting* it, *have all one sign: those made by going round any duad have alternately + and −*. Applying this law generally to all odd and even multiplets, we have the rule: if in a multiplet of n symbols *abcdefghi*...you consider any *continuous portion*, and carry the first letter (or last) of that portion to the place below the last (or above the first) letter of it, you either change the sign of the multiplet *abcd*..., or not, according as the portion considered has an even or an odd number of symbols.

By this rule the signs in the preceding equation are determined; the first six terms, as also the next six, &c. are merely the arrangement (a), if you consider the last three subindices. The first, seventh, thirteenth and nineteenth terms have *alternate signs*, being cyclical permutations of the quadruplet 0123. The subindices of y^2 and x^2 are to be counted as one and the same. It follows from the law of the signs that if for x_0 and y_0 you put any of the three pairs x_1y_1, x_2y_2, x_3y_3, the equation is reduced to a system of self-destroying pairs of terms. Take the third term,

$+(y_0^2+x_0^2)y_2.x_3.1_1$, and let (x_0y_0) be the point (x_3y_3).

The term is now identical in value but not in sign with $(x_3^2+y_3^2)y_2.x_0.1_1$, which stands (the 23 permutation) in the equation with a negative sign, for by our law, 0231 and 3201 have contrary signs, as the first can be transformed into the second by one exchange of a pair, 3 and 0. Or they have contrary signs, if you like, because the first becomes the second by first going one step round 023, thus— 302, which changes no sign, and next going round 02, thus— 320, which changes the sign. In fact you may easily see that *any term* of the (4·3·2·1 =) 24, [62], if you give to any pair of co-ordinates the values of any other pair, (as if you supposed $x_3=x_2$, $y_3=y_2$) will become identical in value, but not in sign, with another term. Thus $(y_2^2+x_2^2).y_1.x_3.1_0$ and $(y_3^2+x_3^2).y_1.x_2.1_0$ are of opposite signs, and destroy each other, if (x_2y_2) and (x_3y_3) be the same point. It is enough, however, if the coincidence of (x_0y_0) with any of the other three points reduces the expression to zero, in order that the equation be true at those points, or that the locus may contain them. As this reduction to zero does thus happen, we are sure that, whatever this locus may be, it passes through (x_1y_1), (x_2y_2), and (x_3y_4), and this too, whatever

co-ordinate angle we employ. As the above equation is of the form

$$(y_0^2 + x_0^2)A + By_0 + Cx_0 + D = 0,$$

it represents a circle, if *our axes are rectangular,* (28) which passes through the three points. The coefficient A is all in the six first terms of (d), and by (51 B) this is exactly $2(\triangle 123)$ the double area of the triangle made by (x_1y_1), (x_2y_3), (x_3y_3). The coefficient B is

$$-\{(y_1^2+x_1^2).(x_2-x_3)+(y_2^2+x_2^3).(x_3-x_1)+(y_3^2+x_3^2).(x_1-x_2)\};\ C \text{ is}$$

$$+\{(y_1^2+x_1^2).(y_2-y_3)+(y_2^2+x_2^2).(y_3-y_1)+(y_3^2+x_3^2).(y_1-y_2)\},$$

and D is

$$-\{(y_1^2+x_1^2).(y_2x_3-y_3x_2)+(y_2^2+x_2^2).(y_3x_1-y_1x_3)$$
$$+(y_3^2+x_3^2).(y_1x_2-y_2x_1)\}.$$

These are all known numbers, so that, as in (28), the centre and the radius can be instantly found, of the first the co-ordinates, and the length of the second. We may write the equation (d) thus, $\{\text{Sin}\ \omega = 1 \text{ in } (51\ \text{B})\}$

$$(\triangle 123)(y_0^2 + x_0^2) - (\triangle 230)(y_1^2 + x_1^2) + (\triangle 301)(y_2^2 + x_2^2)$$
$$-(\triangle 012)(y_3^2 + x_3^2) = 0,$$

a form of little value, except for its symmetry, and for practice in algebraic symbols. Or it may be more briefly symbolized thus,

$$\Sigma(\pm)(y_0^2 + x_0^2).y_1.x_2.1_3 = 0,$$

in which Σ is the token of a *summation* of terms, the terms being all alike, except in their subindices and signs, and all deduced from $+\{(y_0^2+x_0^2).y_1.x_2.1_3\}$ by permutations of the subindices, with their signs determined by the rules above given. The symbol $(\pm)$, bracketted, may denote that the signs have to be properly affixed, after the permutations are written out. To remember the rule for affixing them, you may perhaps find it sufficient aid to say by rote,

[68] Roŭnd ev. or o. with *pin* or no
For signs you go. pin is $+-+-+-$.

i.e. if you cyclically permute any multiplet, *go round* it, you write down the successive permutations *with alternate signs, or no* (i.e. without change) as that multiplet is *even or odd ; pin* for *p*lus m*i*n*u*s.

Thus $+123456 - 162543 + 162345 - 312654$, are four of the $1{\cdot}2{\cdot}3{\cdot}4{\cdot}5{\cdot}6$ permutations of six things, with congruously

determined signs. To find the sign of the second, I take the following steps to reduce the first to that shape,

$$+123456+1(62345)+162(534)-1\acute{6}25(43).$$

As I have gone a step round an odd multiplet, a 5-plet, in the second of these last, I have changed no sign, nor in the next by a step round a triplet, but the next step round a duad changes the sign. The sign of 162345 requires the steps $-12(\acute{6}345)+1(\acute{6}2)345$, the sign of 123456 being twice changed in its transformation; or two operations on -162543 would have sufficed, thus, $+1\acute{6}25(34)+1\acute{6}2(345)$. If we had before us all the permutations of six things, there would be no study at all required after a few steps: you can see that in equation (d) all depends in the correct writing of the first six terms, the rest being three sixes made from those first by *going round* quadruplets—of course with alternate signs.

I may state here, although somewhat too early perhaps for strict method, that a curve of the second degree can always be made to pass through any five points. Its equation is

C. $$\Sigma(\pm)y_0^2.y_1x_1.x_2^2.y_3.x_4.1_5=0.$$

The number of terms is 6 *fags* [62], and if you understand what I have just said about the signs, + or −, of the permutations of six symbols, you can write out the equation yourself; the only difficulty being the length of an operation almost merely mechanical. It matters not what origin or axes be chosen; the curve thus given passes infallibly through every point. Suppose e. g. $x_0=x_5$, $y_0=y_5$, the term

$$+y_0^2.y_1x_1.x_2^2.y_3.x_4.1_5$$

is destroyed by the term

$$-y_5^2.y_1x_1.x_2^2.y_3.x_4.1_0,$$

and the whole equation $=0$, as it ought to be, by similar pairs of internecine terms. What the nature of the curve is, whether a circle or otherwise, will be a delightful enquiry that will ere long occupy us.

You can easily see that, referred to right axes,

$$\Sigma(\pm)(x_0^2+y_0^2)1_1=0$$

is the equation of the circle which has its centre at the origin, and passes through the point (x_1, y_1): that

$$\Sigma(\pm)(x_0^2+y_0^2)y_1=0$$

is that of the circle (29 G) which touches the axis of x at the origin, and passes through $(x_1 y_1)$; and that

$$\Sigma(\pm)(x_0^2+y_0^2)x_1=0$$

touches at the origin the axis of y, and passes also through (x_1, y_1); as also that

$$\Sigma(\pm)x_0 . y_1=0$$

is the equation of the line through the origin and through the point (x_1, y_1). What curves are represented by the above equations when the co-ordinates are not rectangular, is reserved to be examined hereafter.

The equation C contains the terms following:

$$+y_0^2 . y_1x_1 . x_2^2 . y_3 . x_4 . 1_5 - y_1^2 . y_0x_0 . x_2^2 . y_3 . x_4 . 1_5$$
$$+y_1^2 . y_2x_2 . x_0^2 . y_3 . x_4 . 1_5 - y_1^2 . y_2x_2 . x_3^2 . y_0 . x_4 . 1_5$$
$$+y_1^2 . y_2x_2 . x_3^2 . y_4 . x_0 . 1_5 - y_1^2 . y_2x_2 . x_3^2 . y_4 . x_5 . 1_0,$$

which are exactly the following,

$$+y_0^2(y_1x_1 . x_2^2 . y_3 . x_4 . 1_5) - y_0x_0(y_1^2 . x_2^2 . y_3 . x_4 . 1_5)$$
$$+x_0^2(y_1^2 . y_2x_2 . y_3 . x_4 . 1_5) - y_0(y_1^2 . y_2x_2 . x_3^2 . x_4 . 1_5)$$
$$+x_0(y_1^2 . y_2x_2 . x_3^2 . y_4 . 1_5) - 1_0(y_1^2 . y_2x_2 . x_3^2 . y_4 . x_5).$$

Let A stand for $\Sigma(\pm)y_1x_1 . x_2^2 . y_3 . x_4 . 1_5$, the sum of 120 terms made by permutation of subindices only, and let B, C, &c. each stand for such a sum; then the equation is

$$Ay^2 - Byx + Cx^2 - Dy + Ex - F = 0.$$

It is easy to write out completely any of the coefficients: thus D is obtained from $y_1^2 . y_2x_2 . x_3^2 . x_4 . 1_5$ by adding to it 119 terms differing only in subindices. Writing subindices only, $-D$ is

$$\begin{aligned}-\{\ &12345 - 12354 + 12453 - 12435 + 12534 - 12543\\ &- 13452 + 13425 - 13524 + 13542 - 13245 + 13254\\ &+ 14523 - 14532 + 14235 - 14253 + 14352 - 14325\\ &- 15234 + 15243 - 15342 + 15324 - 15423 + 15432\\ &+ 23451 - 23415, \text{\&c.}\}\end{aligned}$$

The same series of permutations of 12345 will give all the six coefficients from their six first terms.

Something further on this mode of writing out explicitly loci, plane or solid, of any class or order, may be seen in a Memoir by the Author *On Linear Constructions*, in the ninth volume N.S. of the *Memoirs of the Philosophical and Literary Society of Manchester*.

JUVENILE CONVERSATIONS.

On §§ 2, 3.

William :—I SEE no difficulty in adding or subtracting negative quantities. Putting on a decrement is taking off something positive; and taking off a decrement is an augmentation, evidently, for it must be the reverse of taking off a positive quantity. It is easy to see that

$$5+(-3) \text{ is } 5-(+3), \text{ the same as } 5-3=2,$$
$$\text{and } 5-(-3) \text{ is } 5+3=8, \text{ vid. [11.]}$$

Between $5+(-3)$ and $5-(-3)$ there is a difference of 6; and while $a+(-b)$ is $=a-b$, $a-(-b)$ is $=a+b$.

But I am puzzled about these co-ordinates. You say that every pair of numbers determines a point; shew me then the point belonging to the numbers 2 and 500.

Richard :—Do you want the point (2,500) or the point (500,2)? Just now, you read for me the points here written,

$$(3, 4)\ (3, -4)\ (-3, 4)\ (-3, -4),$$
$$(4, 3)\ (4, -3)\ (-4, 3)\ (-4, -3);$$

read them again.

William :—The upper line is—the point whose x is 3 and whose y is 4, the point whose x is 3 and whose y is minus 4, the point whose x is minus 3 and whose y is 4, and that whose x is minus 3 and whose y is minus 4. The lower line is—the point whose x is 4 and $y=3$, that whose x is 4 and $y=-3$, and so on: that is the lesson you have taught me. But points should have no magnitude—they cannot be one bigger than another: shew me the points $(3, -4)$ and (2, 500), that I may compare them.

Jane :—I think William does not quite see the use of x and y. Two numbers can determine a point only by measurement; and, before we can measure, we must know from what point and in what directions we are to measure. When you ask Richard for the point (2, 500), you should give him your origin and axes: there is no sense in co-ordinates without these. You must have settled the point O, from which to measure $x=$ two units, (i.e. two inches, on our supposition) along a fixed line OX, and $y=500$ along, or rather in the direction of, the fixed line OY.

William :—Well, then, I draw two lines, and mark them XOX' and YOY': you now take two inches along OX, and 500 inches along OY: pray where is the point (2, 500)? You have found a brace of points, one on each axis!

Richard :—Look again, you philosopher, at my first figure, (vid. p. 3.) I shewed you three ways of coming at the point q, $(3, 2)$: first, your

present method, to take $Op=3$ and $Or=2$, inches, and then to draw through p and r parallels to OY and OX,—these will meet in the point $(3, 2)$: secondly, to take on the axis of x $Op=3$, and at p to raise an ordinate pq parallel to OY, two inches in length; this brings you to the point $(x=3, y=2)$ or $(3, 2)$: thirdly, to measure $y=$ two inches on the axis of y, to r, then from r to draw a parallel to OX, three inches measured on this parallel brings you to the same point q. The second of these methods is the one to bear in mind, when you think of the point $(3, 2)$. Now tell me how you would find $(3, -4)$, and $(2, 500)$.

William:—I first march three inches on OX to p, and thence four inches on an ordinate in the negative direction parallel to OY', this gives me $(3, -4)$: two inches along OX, and then 500 inches along a positive ordinate, bring me to $(2, 500)$.

Jane:—You see it now, and in the same manner you can find your way to the point $\left(\frac{1}{2}, \frac{-5}{7}\right)$ whose x is half an inch positive, and whose y is five-sevenths negative. Did you tell William what we mean by the points

$$(x_1, y_1)\ (-x_1, y_1)\ (x_1, -y_1)\ (-x_1, -y_1)\,?$$

Richard:—By x and y in general we understand the co-ordinates of a variable point, some point or other, no matter for the moment what or where: it is enough for us that x and y are the two distances of a point from two given lines, its distance from each line being measured in a direction parallel to the other line. By x_1 and y_1 we understand the co-ordinates of a fixed point, which we know of and can find. Perhaps x_1 and y_1 are two numbers of which we have thought, or which we have down in a list along with other numbers, marked x_2, y_2, x_3, y_3, &c.; or they are numbers which we can find, when we please, by measurement, of the distances of the given visible point from OX and OY. The points above written are, the point whose x is the known length (x at 1) and whose y is the known number (y at 1), that whose x is the given negative length (x at 1) and whose y, &c. vid. p. 5. The same thing is meant by the point $(x=m, y=n)$, or the point (m, n). We conceive m and n to be known numbers. Suppose now you had before you the given points (m, n) and $(-g, h)$, m, n, g, and h being positive numbers; where would you look for the points $(-m, -n)$ and $(g, -h)$?

William:—I should expect (m, n) to be somewhere between the lines OX and OY, as both the co-ordinates are positive; but $(-g, h)$ is found by proceeding g inches along OX', and the point must be within the angle $X'OY$. As the co-ordinates of $(-m, -n)$ and $(g, -h)$ have signs all contrary to those, the former of these is within the angle $X'OY'$, and the latter within XOY'.

Jane:—True: and you can see that the eight points first written by Richard form the angles of two distinct parallelograms, whether our axes are oblique or rectangular. Have you read, William, the points (x_1, y_1) (x_2, y_2), &c., in (3)? p. 5. There is nothing like reading aloud.

William:—The first is that whose x is $\frac{3}{2}$ and whose y is one; the second that whose x is $\frac{3}{4}$ and whose y is $\frac{1}{2}$, and so on; the fifth (x_5, y_5) has its $x=\frac{3}{5}$ and its $y=\frac{2}{5}$: all easy enough. But how is this, about the point $(0, 0)$ at the end of (3)?

Richard:—Do you not see, from (a'), that if $x=0$, y being $\frac{2}{3}$ of it, is nothing also? How much is $\frac{2}{3}\times 0$?

On § 4.

William:—I am ashamed to say that I stuck fast for a few moments at the pronoun *it*, in the second line of (4). It is nonsense to suppose that OO' can meet the origin a second time. Why must it meet the curve again?

Richard:—It would be a difficult thing to draw a line from O, a point in the supposed curve, which should nowhere meet the curve again; but there is no *must* in the argument: any line you please that cuts the curve in any point O' suffices for the reasoning.

William:—Why did your uncle not set down the lengths in numbers of the co-ordinates of O' instead of saying let them be m and n?

Richard:—I dare say he never measured them: they are considered as known numbers, for the point O' is given and before you. We understand m to be the number of inches in Ob, and n that of those in bO'.

William:—How am I to "find a series of points in the locus (a')"? What is a locus?

Jane:—A *locus* is simply a line, either straight or curved, formed by a series of points arranged in close succession according to a certain *law*. This law is expressed by an *equation*, and this equation is always an assertion about x and y, which is true of the co-ordinates of every point in the locus, and true of no other points. We speak of the line $2y=5x$, $y=ex$, &c: which means, more at length, the line whose law is this or that equation. In the first of these two the law is, that double the y of every point is 5 times the x of it; that of the second is, that y is e times the x at every point of the locus. The point (20, 51) is not in the first line, because it is not true that

$$2\times 51=5\times 20; \text{ but since}$$
$$2\times 50=5\times 20,$$

is truly affirmed, (20, 50) *is* a point of it. If e happens to be the number $2\frac{11}{20}$, (20, 51) is a point of the line $y=ex$, for

$$51=2\tfrac{11}{20}\times 20, \text{ is true;}$$

$$\text{but as}\qquad 50=2\tfrac{11}{20}\times 20, \text{ is false,}$$

(20, 50) is not in this locus.

For a reply to your first question, William, you have only to glance at the process by which a series of points is found in § 3. You may make the same measurements along $O'X''$ and $O'Y''$, that are there made along

OX and OY. Your points so found will have different positions from the former.

William:—I see plainly that my point $(\frac{1}{10}, \frac{1}{15})$ will be near O', while your point $(\frac{1}{10}, \frac{1}{15})$ will be near O. Why is there but one series of points traced in the figure? There are two supposed curves mentioned.

Jane:—You can imagine a second supposed series on the opposite side of OO'; the argument does not require that it should be drawn.

Richard:—Uncle says I was right in calling the argument in § 4 a *reductio ad absurdum*. You see, William, that (a') cannot give a curve when referred to OX and OY, without giving also a *different curve*, when referred to $O'X''$ and $O'Y''$. But from the law (a') by which both curves are traced out, it follows that *any* point of my curve, as $(x = Ot,\ y = st)$, is also a point of yours, viz. the point $(x = O'v,\ y = sv)$; so that the curves are *not different*. The hypothesis which leads to this contradiction must be false; and there is no way of escaping the absurdity, but by supposing that both series of points lie in the right line OO'.

On §§ 5, 6.

Jane:—Cousin Henry shewed me this morning a pretty proof that the product of two negative numbers is positive. We know that

$$(5-5)\times(2-2) = 0\times 0 = 0: \text{ i.e.}$$

$$5\times(2-2) - 5\times(2-2) \text{ must be} = 0, \text{ i.e.}$$

$$0 - 5\times 2 - 5\times(-2) \quad = 0, \text{ i.e.}$$

$$-\ 10\ \ -5\times(-2) \quad = 0, \text{ is true,}$$

which cannot be, unless $-5\times -2 = +10$.

In like manner, $(m-m)$ times $(n-n) = 0\times 0 = 0$,

which requires of necessity that

$$(-m) \text{ times } (-n) = +mn.$$

Divide both these equals by the product $-n\times n$, and you get $-m{:}n = m{:}-n$, as in p. 9.

Is not this charming? I feel quite scientific.

Richard:—Do you not think, Jane, that Article (6) might somehow have been dispensed with? After proving in (4) that $(y = \frac{2}{3}x)$ is a right line, it seems easy to grant, that $y = -\frac{2}{3}x$, or, what is the same, if we multiply both sides by -1, $-y = \frac{2}{3}x$, is a right line also.

Jane:—It is one thing to be convinced of a truth, and another thing to frame a valid proof of it. When you have invented a shorter demonstration than this of (6), I will try to give you my opinion upon it.

William:—I see that the Roman co-ordinates refer to the left hand figure. How do you know that $\mathrm{y} = \frac{2}{3}\mathrm{x}$ is a right line? The axes are different from those in (4). Ah! I see my answer in what precedes here in (6).

Jane:—You observe that the question now is, whether $(y=-\frac{2}{3}x)$ gives points in a right line or no. Whatever this locus may be, two points in it, besides O, can be found; p and m are so found. The question is: are O, p, and m in a right line? Now they form a group exactly similar to O, p, m, in the left-hand figure; and as these are proved in (4) to lie in a right line, O, p, m, are in one also.

On §§ 7, 8.

Richard:—William began to read (7) thus: "We find a point in the locus y which is equal to ex." I laughed, and so did he; but you *must* bring in the *which* in the sentence below—"If qp which is equal to q_1p_1, which is equal to q_2p_2," &c. Uncle says the *which* ought to have been written here; but that the ellipsis is very common in algebraic reasoning, and seldom occasions any ambiguity.

William:—I see that $(y=ex)$ is supposed to be the line q, q_1, q_2 passing through the origin O, which letter you have forgotten to insert, and that p, p_1, p_2 is a line parallel to it, which does not pass through O, but cuts OY and OX'.

Jane:—There are of course innumerable lines parallel to $q, q_1, q_2, \ldots$ some cutting OY and OX', others cutting OY' and OX. The lines $y=ex+1$, $y=ex+2$, $y=ex+b$, if b is positive, all cut the former pair, while $y=ex-1$, $y=ex-2$, $y=ex-b$ all cut the latter; if e has in all one value.

William:—Where is your figure for § 8?

Jane:—Draw it yourself: you may choose any point you like for (x_1, y_1) in any of the four angles about O; this point and O determine a given line. If x_1 and y_1 are both of one sign, as $(2, 3)$, $(-6, -40)$, the line is drawn in the angles XOY and $X'OY'$; if they have different signs, as $(-2, 3)$, $(6, -40)$, the line is drawn in the angles XOY' and $X'OY$. The number of lines through O is infinite; for between any two, however little they may diverge from each other, you can always imagine a line drawn, which diverges still less from either. It is something worth knowing, if it can be proved, that any of those lines through O may be represented by the equation $y=ex$.

Richard:—It appears to me that this is proved in (6); for it there appears, that for every different value of e, positive or negative, that you can name or think of, there is a distinct corresponding line passing through O.

Jane:—True; but is it there demonstrated, that *no line* can pass through O *besides* those which correspond to the positive and negative values which e may receive? This *negative assertion* remains to be proved, and is established in (8). Choose *any* line, so it be a given line through O. If it is given, a point of it is known besides O. Give me this point, i.e. give me its co-ordinates in numbers, and I will write down the equation of *a* line, as it is done in (8), and make you confess that it is *your* line, and that it has an equation of the form asserted.

William:—It is evident, looking at the second proposition in (8), that a line which does not pass through O must meet YOY' somewhere, at some distance b, positive or negative. But is this proposition true of *every* line, of lines through O also?

Richard:—Of course, if b may have *any* value: for lines through O, $b=0$.

On §§ 9, 10, 11.

William:—I can make nothing of your equation (c'), p. 12. How do you read it?

Jane:—Did you never handle a fraction whose denominator was a fraction? If my master had taught you arithmetic! Look at the middle of the next page—you see y divided by the fraction $\frac{2}{7}$, and the negative quotient x divided by the fraction $\frac{8}{21}$. You should always read an argument through, and all the illustrations, before you despair. I read (c') thus: "the negative quotient x by the fraction (b by e), $+y$ by b, $=1$;" or, if you like, "y by b, minus x by (b by e), $=1$."

When $y=0$, which is true of every point in XOX', this becomes

$$-\frac{x}{\frac{b}{e}}=1,$$

or since $-\frac{m}{n}=\frac{m}{-n}$, (5),

$$\frac{x}{-\frac{b}{e}}=1,$$

whence, as the denominator and numerator must be equal, we have $x=-\frac{b}{e}$.

Richard:—The example at the end of the Lesson, (p. 15), became much clearer to me, after I had drawn the figure, and examined the case of

$$y=2x+1 \quad \text{and} \quad y=-2x+1, \quad \text{which are}$$

$$\frac{y}{1}+\frac{x}{-\frac{1}{2}}=1 \quad \text{and} \quad \frac{y}{1}+\frac{x}{\frac{1}{2}}=1.$$

I took $n=3$ and obtained $y_1=6+1=7$, $y_2=-6+1=-5$; whence $y_1+y_2=7-5=2$, or $PB-PC=2.PA$, QA being the parallel to OX through $(0, 1)$, or Q. Now from

$$PB-PC=PA+PA, \quad \text{just proved,}$$

comes
$$PB=PA+PA+PC,$$

by addition of $+PC$ to the equals; and hence we get

$$PB-PA=AP+PC,$$

or
$$AB=AC,$$

by subtraction of PA from the equals.

Thus I prove that QA always bisects the base of the triangle QBC, whatever ordinate $BAPC$ may be.

Jane:—Well demonstrated, Richard; but you speak as if you had found all that out yourself. I will now shew William something more, the discovery of which he will of course ascribe entirely to me.

The line through $(0, b)$ parallel to OX, spoken of, p. 15, has for its equation $y=b$, or $y-b=0$; for at every point of it, the ordinate is of the same length b; and every line parallel to OX can be represented by the same equation, for some fixed value or other of b; thus $y=-b$, or $y+b=0$ is the equidistant parallel on the other side of OX, through the point $(0, -b)$. Every parallel to OY has an equation of the form $x=b$; thus $x-4=0$, and $x+4=0$, cut OX and OX' at equal distances from O, and are both parallel to OX, because x never changes in either line, however y may vary. The equation of the ordinate drawn in your figure through P is $x-3=0$; and every ordinate can be represented by $x-n=0$ for some value positive or negative of n.

The property you have just proved may be stated thus: the line $y-b=0$, (QA), bisects that portion of the line $x-n=0$ $(BAPC)$, which lies between the lines $y-ex-b=0$ and $y+ex-b=0$; whatever be the numbers e or n. This is true of course when $b=0$, in which case the bisecting line QA is $y=0$ or OX. The two lines are now $y=ex$ and $y=-ex$, and we know something worth remembering about the four lines $y=0$, $x=0$, $y-ex=0$, $y+ex=0$, which all meet in O.

Richard:—There could hardly be a simpler combination; there is only one constant in the four equations, the number e, used with opposite signs.

Jane:—Uncle Penyngton remarked that simple combinations are the most instructive, and lead, when their meaning is completely mastered, to the most extensive and entertaining views. The thing to be remembered is, that the first of these four lines bisects every parallel to the second drawn from the third to the fourth; and that the second bisects every parallel to the first, which is drawn from the third to the fourth. Thus let $y=m$ in both the latter, which will be the case where each meets the line $y-m=0$; the equations become

$$m = ex_1,$$

$$m = -ex_2, \text{ whence by subtraction,}$$

$$m - m = ex_1 - (-ex_2), \text{ which is}$$

$$0 = ex_1 + ex_2, \text{ or since } 0 \div e = 0,$$

$$0 = x_1 + x_2, \text{ by div. of equals by } e,$$

$$\text{i.e.} \quad -x_1 = x_2;$$

or, the x of the point on $(y-ex=0)$ whose y is m, differs from the x of the point on $(y+ex=0)$, whose y is m, only in sign. In other words, OY bisects the parallel to OX, which is drawn between those two lines. This is true whatever m may be, and is not affected by the value of e;

neither does it depend on the angle XOY between $x=0$ and $y=0$, any more than on that between $y-ex=0$, and $y+ex=0$. I am told that these four lines form *an harmonic pencil*, and that such a pencil has many curious properties which we shall be delighted to learn.

On §§ 12, 13.

William:—How can anybody be expected to remember such a long piece of reasoning as this in the 12th Article? I am utterly confused in it.

Richard:—So it was at first with both Jane and myself; but it is pleasant to find how rapidly such difficulties melt away, after a little examination. I think it all through at a glance, when I look at the figure in (13).

First of all I know that if $y{:}x=e$ be any line through the origin, no matter what be the angle between the axes OX and OY, containing the points (x_1, y_1), (x_2, y_2), (x_3, y_3), &c.,

$$\frac{y_1}{x_1}=\frac{y_2}{x_2}=\frac{y_3}{x_3}=\frac{y_4}{x_4}=\&\text{c.}$$

because each of these fractions is one number e. From the first pair, multiplying the equals by $\frac{x_1}{y_2}$, comes

$$\frac{y_1}{y_2}=\frac{x_1}{x_2}.$$

This is equation B′ (p. 17), if OY and OX are my two axes, and OP the line through the origin, and it is equation b′, if OY and OP are my axes, and OX' the line through the origin. The two ordinates y_1 and y_2 are the same, whether OX or OP be the axis of x, except that in the first case they are both positive, qp and QP, in the direction OY, and in the latter both negative, pq and PQ, in the direction YO. This makes no difference in the truth of the last equation, since $-y_1{:}-y_2=y_1{:}y_2$. Thus I have proved the third line of [6]; for the *parall. cutters* are the two ordinates y_1 and y_2, and the quotient of these is that of the two abscissæ, whether these be Op and OP, or Oq and OQ. The whole argument of (12) is in the equations B′, b′, and the equation Bb; the last deduced from B and b by division, the former pair by multiplication. The second line of [6] may be considered to assert the equality of the right members of B′ and b′, as well as to express (Bb).

On § 14.

William:—He was puzzled at 'turned face downwards?' If the triangle ADC is turned over in its place without disturbing A, where can C fall but on C', or D but on D'? Multiplication of a line by a line is not so easy a notion—how can you multiply but by a number?

Jane:—You see that *line times line* appears in the equations: this

cannot be the same thing with *number times line*, which is just a longer line. We must accept the definition: and it is easy to conceive an inch square to be made by the repetition of an inch line through a height equal to its length, and to call such a square pile of inch lines, an *inch times an inch.*

A quarter-inch repeated to a height equal to half an inch, is $\frac{1}{4} \times \frac{1}{2} = \frac{1}{8} \times$ a square unit; and so of any two lines.

On § 16.

Richard:—The numbers x_1, x_2, y_1, y_2, eq. (8) are lengths measured from the two visible points (x_1, y_1) (x_2, y_2) to our axes OX and OY. Suppose now that we had chosen other axes, inclined at any particular angle: these numbers would be different, some or all of them; but the position of the points is fixed, and therefore the position of the line through them; while the equation of the line alters for every different pair of axes. Is it not odd, and confusing to imagine, that the same visible line should be represented and determined by so many different equations?

Jane:—Just as odd, as that the same point in our plane should be represented and determined by innumerable pairs of numbers, i.e. by a different pair for every different pair of axes. But we use only one pair OX, OY, in one argument; we know always what x and y stand for, and how they are measured. Don't you think that this is the chief beauty of equation (8), that it applies to any axes we may choose? Is it not worth knowing, that, when they are chosen, whatever be the distances x_1, x_2, y_1, y_2, that equation is sure to be true of every point (x_0, y_0) in a line with (x_1, y_1) and (x_2, y_2)? Thus, if $x_1 = 0 = y_1$, it takes the shape $y - ex = 0$, or rather $Ax + By = 0$,—the same thing, as to the form of the equation.

On § 21.

William:—How can you say that $p_2m_2 = m_2q_2 + q_2p_2$? Is not the distance from q_2 to p_2 negative, q_2p_2 being a negative ordinate?

Richard:—You may take it thus: $+p_2m_2 = +p_2q_2 + q_2m_2$; we are measuring from p_2 to m_2 in the direction OY, which is positive: and if you state it thus: $-m_2p_2 = -m_2q_2 - q_2p_2$, putting negative signs because we measure now from m_2 to p_2, it comes to the same thing; for multiplying both sides of this equation by -1, you obtain the former equation. It is useful to remember that *you may change the signs of every term on both sides of any equation, without altering the truth of it*; for this is merely multiplying equals by -1. If one side of the equation, as it frequently happens, is zero, you may change the sign of every term on the other side. Thus

$$5 - 2 = 3, \qquad -5 + 2 = -3,$$
$$5 - 2 - 3 = 0, \qquad -5 + 2 + 3 = 0,$$

are all equally true. It is evident that $0 \times -1 = 0$, for $+0$ and -0 are the same quantity, at least in arithmetic value. I thought it an odd and

mysterious remark of Uncle Penyngton, which fell from him the other day, that if ever I mounted into the higher regions of analysis, I might learn to make important distinctions between positive and negative zero!

William:—I am not satisfied about the two equations $y_2 = -q_2p_2$, and $y_3 = -q_3p_3$. It is plain to be seen that y_2 is negative, and so is y_3, and that y_2 and y_3 are nothing else than the two lines q_2p_2 and q_3p_3; why then do you not write $-y_2 = -q_2p_2$, and $-y_3 = -q_3p_3$?

Jane (after a pause):—This is very acute of you; and it has cost me some thinking to find a sufficient answer to your objection. But I have it now: in the equations with which (21) begins, the co-ordinates are of course numbers of inches, l is *the number* l, a is *the number* a, and Oq_1 p_1q_1, p_2q_2, &c., are put for the *numbers* of inches in those lines. We must read, let p_2 be the point whose x is the positive number (of inches in) Oq_2, and whose y is the number (of inches in) p_2q_2 taken negatively.

If we were to write, as you propose,

$$-y_2 = -q_2p_2, \qquad -y_3 = -q_3p_3,$$

it would follow immediately that

$$y_2 = q_2p_2, \text{ and } y_3 = q_3p_3,$$

making y_2 and y_3 to be positive ordinates, which they are not.

On § 21, 23.

William:—You ask (p. 35) for the distance between (2, 3) and (·3 —·4) referred to right axes. Would the distance be different if the axes were oblique?

Richard:—Of course not: *referred* here agrees with *points*. But $(x-b)^2+(y-a)^2$ is not the squared distance between (x, y) and (b, a), if these co-ordinates are measured parallel to oblique axes. I do not exactly see why not, at this moment; but I do see the proof of the rule (p. 36) laid down in [12]. The argument leaves the choice of axes to me. Provided that they are at right angles, and the same axes all through the reasoning, it matters not where they are. This is to me the great beauty, as Jane says, of equation D′ and the rule under it, that R is sure to come out the right distance between (x_1, y_1) and (x_0, y_0), whatever be the right axes OX and OY, from which co-ordinates are measured.

Jane:—The reason of this is, that R depends not exactly on the distances x_1, x_0, y_1, y_0, but on the algebraic differences of distances, (x_1-x_0) and (y_1-y_0). If x_1 and x_0 are the same number, R becomes y_1-y_0. The two points are now on the same ordinate, and on the same side or opposite sides of OX, according as y_1 and y_0 are numbers of like or unlike signs.

William:—I can see all this now: and am delighted with my introduction to the equation of the circle, which is, like the right line, a locus of points arranged according to a law laid down in the equation.

Richard:—The first difference to be observed between the line and the circle is, that in the former there is obtained from the equation but one

value of y corresponding to any particular value of x; while in the latter you obtain two values of y for every value that you introduce for x, and two of x for every given value of y: a pair of possible, or else a pair of impossible, values.

William :—This is made clear enough in (23). The circle in page 39 can be written thus

$$\{x-(-5)\}^2+(y-3)^2=(0{\cdot}7)^2,$$

exactly of the form (D, 21).

On §§ 28, 29.

Richard :—You see, William, that we can *calculate* a radius to the ten-millionth of an inch, if we have the equation to the circle. You could not *measure* it, on the figure, to a thousandth.

Let me now see you try, after what I have said to you, what you can make of the equation

$$ax^2+ay^2+bx+cy+d=0,$$

the axes being of course rectangular.

William :—As x^2 and y^2 have the same coefficient, I am to consider this to be a circle, as I suppose; but I ought to be able to *prove* it a circle. How is this to be done?

Richard:—If you can shew that it is of the form (D, 21,) you will prove it completely.

William :—Let me try to find the centre and the radius by the process of this Art. 28. Transposing d, we have

$$ax^2+ay^2+bx+cy=-d,$$

then adding to the equals the squares of the half coefficients of x and y,

$$ax^2+ay^2+bx+cy+\frac{b^2}{4}+\frac{c^2}{4}=-d+\frac{b^2}{4}+\frac{c^2}{4};$$

which should fall into the form (D, 21) thus,

$$\left(ax^2+bx+\frac{b^2}{4}\right)+\left(ay^2+cy+\frac{c^2}{4}\right)=\frac{b^2}{4}+\frac{c^2}{4}-d.$$

Now I can see that this comes right, if $a=1$, for it is then

$$\left(x^2+bx+\frac{b^2}{4}\right)+\left(y^2+cy+\frac{c^2}{4}\right)=\left(\sqrt{\frac{b^2}{4}+\frac{c^2}{4}-d}\right)^2$$

$$\text{i.e. } (x+\tfrac{1}{2}b)^2+(y+\tfrac{1}{2}c)^2=(\sqrt{b^2{:}4+c^2{:}4-d})^2,$$

the circle whose centre is $(-\frac{1}{2}b,\ -\frac{1}{2}c)$, and radius $=\sqrt{b^2{:}4+c^2{:}4-d}$.

Richard :—You forgot to divide all at first by the common coefficient of x^2 and y^2; if you had done this, it would have been correct after adding to both sides the squares of $\frac{1}{2}b{:}a$ and $\frac{1}{2}c{:}a$. You would have had

$$x^2+y^2+\frac{b}{a}x+\frac{c}{a}x=-\frac{d}{a}$$

$$\left(x^2+2\cdot\frac{b}{2a}x+\frac{b^2}{4a^2}\right)+\left(y^2+2\cdot\frac{c}{2a}y+\frac{c^2}{4a^2}\right)=-d+\frac{b^2}{4a^2}+\frac{c^2}{4a^2},$$

$$\text{i.e. } (x+\tfrac{1}{2}b:a)^2+(y+\tfrac{1}{2}c:a)^2=(\sqrt{b^2:4a^2+c^2:4a^2-d:a})^2,$$

the circle whose centre is $(-\frac{1}{2}b:a, -\frac{1}{2}c:a)$ and radius

$$=\sqrt{b^2:4a^2+c^2:4a^2-d:a}.$$

If $a=3$, $b=5$, $c=6$, $d=-7$, happen to be the values of the four constants, this is exactly the circle (E). Examine next

$$2x^2+20x+2y^2-12y+67\cdot02=0.$$

William :—The steps are easy when all the coefficients are numerical. I have, dividing equals by 2,

$$x^2+10x+y^2-6y+33\cdot51=0,$$

$$(x^2+2.5x+5^2)+(y^2-2.3y+3^2)=5^2+3^2-33\cdot51,$$

$$\text{i.e. } (x+5)^2+(y-3)^2=0\cdot49,$$

the circle figured at page 39.

Richard :—I have made various examples for myself and reduced them: all you have to do is to add the two proper squares, and then reduce by [14]. Take the pair of circles,

$$7x^2+7y^2+8y=0, \qquad ax^2+ay^2-5x=0;$$

and try to prove by (29), that the first touches OX at O, on the under or negative side, having the radius $=\frac{4}{7}$, and that the latter touches OY on the right or positive side, at the distance $5:2a$.

On §§ 32, 34.

Richard :—The circular functions are soon fixed in the memory, when you look at them in the figure, p. 55. The arc being AP, Pp is the Sine, Op the cosine, AT the tangent, OT the secant, DR the cotangent, OR the cosecant, AP the versed sine, of the arc; i.e. the numbers θ, $\sin\theta$, $\cos\theta$, &c., are exactly the lengths in inches of those lines, if OA is unity, or one inch.

William :—Some arcs are greater than a right angle; where do you look for the complement (Def. 6) of such an arc.

Jane :—The complement must be of course a negative arc, measured from A to some point below OA. And you will find, if you examine, that the x and y of a point a certain distance beyond D towards B, are exactly the y and x of a point the same distance below A towards R; so that G, p. 59, is true in the case of ω negative also.

Richard :—I see that in (34) the segments of the side AB are measured from D to its extremities, whether D is in AB or AB produced.

It is pleasant to have Euclid II. 12 and 13, in so small a compass in [26 A].